Journal of Mathematical Research & Exposition
Vol.22, No.3, 337-344, August, 2002

Periodic Boundary Value Problems for Second Order Integro-Differential Equations in Banach Spaces *

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Abstract: This paper investigates the maximal and minimal solutions of periodic boundary value problems for second order integro-differential equations in Banach spaces by establishing a comparison result and using the monotone iterative method.

Key words: Ordered Banach space; maximal and minimal solutions; periodic boundary value problems.

Classification: AMS(2000) 34B10/CLC O175.6

Document code: A **Article ID:** 1000-341X(2002)03-0337-08

1. Introduction

The use of monotone iterative methods in the study of the periodic boundary value problems for integro-differential equations in Banach spaces has recently been quite extensive (see, for example [1–4]). The paper [1] investigates the existence of maximal and minimal solutions of periodic boundary value problems for first order integro-differential equations in real number spaces by establishing a comparison result and using the upper and lower solutions. These methods have been extended and improved and more profound results have been obtained in [2]. In this paper, we shall present the comparison result for second order problems in Banach spaces, and investigate the existence of maximal and minimal solutions of periodic boundary value problems for second order integro-differential equations in Banach spaces.

Let $(E, |\cdot|)$ be a Banach space which is partially ordered by a cone P of E, in this paper we shall consider the following second order periodic boundary value problem (PBVP):

$$\begin{cases} u'' = f(t,u,Tu), & t \in J \ \text{a.e.}, \\ u(0) = u(a), \ u'(0) = u'(a), \end{cases} \tag{1}$$

*Received date: 1999-12-03
Foundation item: Supported by Natural Science Foundation of Hainan Province (10102)
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where, $f \in C(J \times E \times E, E), J = [0, a]$ for $a > 0$, the mapping $T : E \to E$ be defined by

$$(Tx)(t) = \int_0^t K(t,s)x(s)\mathrm{d}s, \tag{2}$$

with $K \in C(D, R_+)$. Let $D = \{(t,s) \in J \times J : t \geq s\}$, and $S_0 = \max_D K(t,s)$ (then we have that $S_0 \geq 0$). The function $u \in C^2(J, E)$ is called a solution of PBVP(1), if u satisfies equation (1). Throughout this paper we always assume that $x \leq y$ if and only if $x(t) \leq y(t)$ for every $t \in J$ and any $x, y \in C(J, E)$.

2. Hypotheses and Auxiliary Results

Lemma 1 (comparison results) *Assume that $p \in C^1(J, E)$ satisfies*

$$p'' \leq -Mp - NTp - r_p, \tag{3}$$

where $M > 0$ and $N \geq 0$ satisfy that

$$a^2M + a^3S_0N < \frac{1}{2} \tag{4}$$

and

$$r_p = \begin{cases} 0 & \text{if } p(0) \leq p(a) \text{ and } p'(0) \leq p'(a), \\ \beta_p & \text{if } p(0) > p(a) \text{ and } p'(0) > p'(a), \end{cases}$$

with

$$\beta_p = \begin{cases} (\frac{Mt^2+2}{a^2} + \frac{NS_0t^3}{3a^2})[(p(0) - p(a)) + (p'(0) - p'(a))] & \text{if } a \leq 2, \\ (\frac{Mt^2+2}{a^2} + \frac{NS_0t^3}{3a})[(p(0) - p(a)) + (p'(0) - p'(a))] & \text{if } a > 2, \end{cases}$$

then $p(t) \leq 0$ for $t \in J$.

Lemma 2 *Assume that $p \in C^1(J, E)$ satisfies*

$$p'' \geq Mp + NTp + \eta_p, \tag{5}$$

where $M > 0$ and $N \geq 0$ satisfy (4) and

$$\eta_p = \begin{cases} 0 & \text{if } p(0) \geq p(a) \text{ and } p'(0) \geq p'(a), \\ \alpha_p & \text{if } p(0) < p(a) \text{ and } p'(0) < p'(a), \end{cases}$$

with

$$\alpha_p = \begin{cases} \frac{M(a-t)^2+2}{a^2} + \frac{NS_0(a-t)^3+NS_0a^3}{3a^2})[(p(a) - p(0)) + (p'(a) - p'(0))] & \text{if } a \leq 2, \\ \frac{M(a-t)^2+2}{a} + \frac{NS_0(a-t)^3+NS_0a^3}{3a})[(p(a) - p(0)) + (p'(a) - p'(0))] & \text{if } a > 2, \end{cases}$$

then $p(t) \leq 0$ for $t \in J$.

The proof of the above two lemmas is omitted, similar results we see [5].

Lemma 3 *The function $\mu \in C^2(J, E)$ is the solution of the PBVP(1) if and only if the* *function x, where $x=(x_1,x_2)=(\mu,\mu')$, is solution of the operator equation $Ax=x$, the components of $A=(A_1,A_2)$ being defined by*

$$(A_1x)(t)=\frac{e^{-\lambda t}}{e^{\lambda a}-1}\int_0^a e^{\lambda s}[\lambda x_1(s)+x_2(s)]\mathrm{d}s+e^{-\lambda t}\int_0^t e^{\lambda s}[\lambda x_1(s)+x_2(s)]\mathrm{d}s, \tag{6}$$

$$\begin{aligned}(A_2x)(t)=&\frac{e^{-\lambda t}}{e^{\lambda a}-1}\int_0^a e^{\lambda s}[\lambda x_2(s)+f(s,x_1,Tx_1)]\mathrm{d}s+\\ &e^{-\lambda t}\int_0^t e^{\lambda s}[\lambda x_2(s)+f(s,x_1,Tx_1)]\mathrm{d}s,\end{aligned} \tag{7}$$

where $\lambda>0$ is arbitrarily given constant.

Defining $\alpha=\begin{cases} a^2 & \text{if } a\le 2\\ a & \text{if } a>0\end{cases}$. Let us impose the following hypotheses on the function f:

(H$_1$) (i) There exist u_0 and $v_0\in C^2(J,E)$ such that $u_0\le v_0$ and $u_0''\le f(t,u_0,Tu_0)-\gamma_{u_0}$, $v_0''\ge f(t,v_0,Tv_0)+\gamma_{v_0}$, where $\gamma_{u_0},\gamma_{v_0}$ staisfy respectively that

$$\gamma_{u_0}=\begin{cases}0 \quad \text{if } u_0(0)\le u_0(a),u_0'(0)\le u_0'(a),\\ (\frac{Mt^2+2}{\alpha}+\frac{NS_0t^3}{3\alpha})[(u_0(0)-u_0(a))+(u_0'(0)-u_0'(a))]\\ \quad \text{if } u_0(0)>u_0(a),u_0'(0)>u_0'(a),\end{cases}$$

$$\gamma_{v_0}=\begin{cases}0 \quad \text{if } v_0(0)\ge v_0(a),v_0'(0)\ge v_0'(a),\\ (\frac{Mt^2+2}{\alpha}+\frac{NS_0t^3}{3\alpha})[(v_0(0)-v_0(a))+(v_0'(0)-v_0'(a))]\\ \quad \text{if } v_0(0)<v_0(a),v_0'(0)<v_0'(a),\end{cases}$$

with $M>0,N\ge 0$.

(ii) There exist u_0 and $v_0\in C^2(J,E)$ such that $v_0\le u_0$ and $u_0''\le f(t,u_0,Tu_0)-\overline{\gamma}_{u_0}$, $v_0''\ge f(t,v_0,Tv_0)+\overline{\gamma}_{v_0}$, where $\overline{\gamma}_{u_0},\overline{\gamma}_{v_0}$ satisfy respectively that

$$\overline{\gamma}_{u_0}=\begin{cases}0 \quad \text{if } u_0(0)\le u_0(a),u_0'(0)\le u_0'(a),\\ (\frac{M(a-t)^2+2}{\alpha}+\frac{NS_0(a-t)^2+NS_0a^3}{3\alpha})[(u_0(0)-u_0(a))+(u_0'(0)-u_0'(a))]\\ \quad \text{if } u_0(0)>u_0(a),u_0'(0)>u_0'(a),\end{cases}$$

$$\overline{\gamma}_{v_0}=\begin{cases}0 \quad \text{if } v_0(0)\ge v_0(a),v_0'(0)\ge u_0'(a),\\ (\frac{M(a-t)^2+2}{\alpha}+\frac{NS_0(a-t)^2+NS_0a^3}{3\alpha})[(v_0(0)-v_0(a))+(v_0'(0)-v_0'(a))]\\ \quad \text{if } v_0(0)<v_0(a),v_0'(0)<v_0'(a),\end{cases}$$

with $M>0,N\ge 0$.

(H$_2$) (i) $f(t,u,v)-f(t,\bar u,\bar v)\ge -M(u-\bar u)-N(v-\bar v)$ for $u_0\le\bar u\le u\le v_0,Tu_0\le\bar v\le v\le Tv_0$;

(ii) $f(t,u,v)-f(t,\bar u,\bar v)\le M(u-\bar u)+N(v-\bar v)$ for $v_0\le\bar u\le u\le u_0,Tv_0\le\bar v\le v\le Tu_0$.

(H$_3$) Assume $M>0,N\ge 0$ satisfy (4), and there exists a constant $L>0$ such that

$$\begin{aligned}b_1=&\frac{La}{1-e^{-La}}+\frac{NS_0a^2e^{La}}{L(1-e^{-La})}+\frac{(L+M)a^2}{(1-e^{-La})^2}<1,\\ b_2=&\frac{(L+M)a}{1-e^{-La}}+\frac{NS_0ae^{La}}{L}+(1+b_1)L<1.\end{aligned}$$

Take $u_0, v_0 \in E$ with $u_0 < v_0$, we call $[u_0, v_0] = \{x \in E : u_0 \le x \le v_0\}$ an ordered interval of E.

3. Main Results

Now, we give the following two existence theorems, which are main results of this paper.

Theorem 1 *Let cone P be regular. Assume that (H_1) (i), (H_2) (i) and (H_3) hold, then, there exists monotone sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ which converge in $C^1(J,E)$ to the minimal and maximal solutions x, y of PBVP(1) in $[u_0, v_0]$.*

Theorem 2 *Let cone P be regular. Assume that (H_1) (ii), (H_2) (ii) and (H_3) hold, then, there exists monotone sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ which converge in $C^1(J,E)$ to the minimal and maximal solutions x, y of PBVP(1) in $[v_0, u_0]$.*

We shall prove Theorem 2, because the proof of Theorem 1 is similar.

The proof of Theorem 2 (I) For any $w \in [v_0, u_0]$, consider the PBVP for the linear integro-differential equation

$$u'' = Mu + NTu + z(t), u(0) = u(a), u'(0) = u'(a) \tag{8}$$

where $z(t) = f(t, w(t), (Tw)(t)) - Mw(t) - N(Tw)(t)$. We shall show that there exists a unique solution of PBVP(8), for each $w \in [v_0, u_0]$. From Lemma 3 it follows that PBVP(8) is equivalent to the following operator equation:

$$\begin{cases} u' = A_2\bar{x} \triangleq \overline{A}\mu = \dfrac{e^{-Lt}}{e^{La}-1}\displaystyle\int_0^a e^{Ls}[Lu'(s) + M\mu(s) + N(T\mu)(s) + z(s)]\mathrm{d}s + \\ \qquad\qquad e^{-Lt}\displaystyle\int_0^t e^{Ls}[L\mu'(s) + M\mu(s) + N(T\mu)(s) + z(s)]\mathrm{d}s, \\ \mu(0) = \mu(a), \end{cases} \tag{9}$$

where $\bar{x} = (\mu, \mu')$, L is given by (H_3). It is easy to see that (9) is equivalent to the following equation: $\mu(t) = (S\mu)(t)$ where the operator S is defined by

$$(S\mu)(t) = \frac{e^{-Lt}}{e^{La}-1}\int_0^a e^{Ls}[(\overline{A}\mu(s) + L\mu(s)]\mathrm{d}s + e^{-Lt}\int_0^t e^{Ls}[(\overline{A}\mu(s) + L\mu(s)]\mathrm{d}s. \tag{10}$$

Therefore

$$(S\mu)'(t) = (\overline{A}\mu)(t) + L\mu(t) - L(S\mu)(t) \tag{11}$$

Defining a norm $\|\cdot\|$ of $C(J,E)$ such that $\|u\| = \max_J |u(t)e^{Lt}|$, $\|\cdot\|_1$ of $C^1(J,E)$ such that $\|u\|_1 = \max\{\|u\|, \|u'\|\}$, we show that S is a contraction in $C^1(J,E)$. In fact, for any u and $u' \in C^1(J,E)$, we have

$$\|\overline{A}\mu - \overline{A}\overline{\mu}\| = \max_J |[(\overline{A}\mu)(t) - (\overline{A}\overline{\mu})(t)]e^{Lt}|$$

$$\le \frac{1}{e^{La}-1}\left|\int_0^a e^{Ls}[Lu'(s) + M\mu(s) + N(T\mu)(s) - L\overline{\mu}'(s)M\overline{\mu}(s) - N(T\overline{\mu})(s)]\mathrm{d}s\right| +$$

$$\max_J \left|\int_0^t e^{Ls}[L\mu'(s) + M\mu(s) + N(T\mu)(s) - L\overline{\mu}'(s) - M\overline{\mu}(s) - N(T\overline{\mu})(s)]\mathrm{d}s\right|$$

$$
\begin{aligned}
&\le \frac{e^{La}}{e^{La}-1}\int_0^a e^{Ls}[L|\mu'(s)-\overline{\mu}'(s)|+M|\mu(s)-\overline{\mu}(s)|+|N(T\mu)(s)-(T\bar{)}\mu(s)|]ds\\
&\le \frac{(L+M)ae^{La}}{e^{La}-1}\|\mu-\overline{\mu}\|_1+\frac{Ne^{La}}{e^{La}-1}[\int_0^a e^{Ls}|\int_0^s K(s,\xi)(\mu(\xi)-\overline{\mu}(\xi))d\xi|ds]\\
&\le \frac{(L+M)a}{1-e^{-La}}\|\mu-\overline{\mu}\|_1+\frac{NS_0}{1-e^{-La}}\int_0^a e^{Ls}|\int_0^s e^{-L\xi}d\xi|ds\|\mu-\overline{\mu}\|_1\\
&\le [\frac{(L+M)a}{1-e^{-La}}+\frac{NS_0ae^{La}}{L}]\|\mu-\overline{\mu}\|_1.
\end{aligned}
$$

Let $b_3=\frac{(L+M)a}{1-e^{-La}}+\frac{NS_0ae^{La}}{L}$, then,

$$
\begin{aligned}
\|S\mu-S\overline{\mu}\| &= \max_J |[(S\mu)(t)-(S\overline{\mu})(t)]e^{Lt}|\\
&\le \frac{1}{e^{La}-1}|\int_0^a e^{Ls}[(\overline{A}\mu)(s)+L\mu(s)-(\overline{A}\overline{\mu})(s)-L\overline{\mu}(s)]ds|+\\
&\quad \max_J|\int_0^t e^{Ls}[(\overline{A}\mu)(s)+L\mu(s)-(\overline{A}\overline{\mu})(s)-L\overline{\mu}(s)]ds|\\
&\le \frac{e^{La}}{e^{La}-1}\int_0^a[|((\overline{A}\mu)(s)-(\overline{A}\overline{\mu})(s))e^{Ls}|+L|(\mu(s)-\overline{\mu}(s))e^{Ls}|]ds\\
&\le \frac{1}{-e^{-La}+1}\int_0^a[\|\overline{A}\mu-\overline{A}\overline{\mu}\|+L\|\mu-\overline{\mu}\|]ds\\
&\le \frac{ab_3}{1-e^{-La}}\|\mu-\overline{\mu}\|_1+\frac{La}{1-e^{-La}}\|\mu-\overline{\mu}\|_1\\
&= b_1\|\mu-\overline{\mu}\|_1.
\end{aligned}
$$

From (11) we have

$$
\begin{aligned}
\|(S\mu)'-S(\overline{\mu})'\| &\le \|\overline{A}\mu-\overline{A}\overline{\mu}\|+L\|\mu-\overline{\mu}\|+L\|S\mu-S\overline{\mu}\|\\
&\le b_1\|\mu-\overline{\mu}\|_1+\|\mu-\overline{\mu}\|_1+Lb_1\|\mu-\overline{\mu}\|_1\\
&= b_2\|\mu-\overline{\mu}\|_1.
\end{aligned}
$$

By assumption (H_3), it follows that $b_1<1$ and $b_2<1$. Therefore, $\|S\mu-S\overline{\mu}\|_1<\|\mu-\overline{\mu}\|_1$, i.e. S is a contraction. By virtue of Banach's fixed point theorem, there exists a unique fixed point $u\in C(J,E)$ of S such that u is a unique solution of PBVP(8) for each $w\in[v_0,u_0]$.

(II) We define a mapping $Bw=u$ for any $w\in[v_0,u_0]$, where u is the unique solution of PBVP(8) relative to the w. We show that

(1^0) $v_0\le Bv_0$ and $u_0\ge Bu_0$;

(2^0) B possesses a monotone nondecreasing property on the segment $[v_0,u_0]$, that is, if $z,w\in[v_0,u_0]$ with $w\le z$ then $Bw\le Bz$.

In order to prove (1^0), let $v_1=Bv_0$ and $p=v_0-v_1$, then $p(0)-p(a)=v_0(0)-v_0(a), p'(0)-p'(a)=v_0'(0)-v_0'(a)$, and

$$
\begin{aligned}
p''(t)=v_0''(t)-v_1''(t) &\ge f(t,v_0,Tv_0)+\overline{\gamma}_{v_0}-Mv_1-NTv_1+Mv_0-NTv_0-f(t,v_0,Tv_0)\\
&= Mp(t)-N(Tp)(t)+\overline{\gamma}_p,
\end{aligned}
$$

where

$$\overline{\gamma}_p = \overline{\gamma}_{v_0} = \begin{cases} 0 & \text{if } p(0) \geq p(a) \text{ and } p'(0) \geq p'(a), \\ \alpha_p & \text{if } p(0) < p(a) \text{ and } p'(0) < p'(a), \end{cases}$$

with α_p as in Lemma 2. By means of Lemma 2 and the assumption (H_3) we get $p(t) \leq 0$ on J, therefore, $v_0 \leq Bv_0$. Similarly, we can prove that $u_0 \geq Bu_0$.

In order to show (2^0), let $w, z \in [v_0, u_0]$ with $w \leq z$, set $\eta = Bw, \mu = Bz, p = \eta - \mu$. By assumption ($H_2$)(ii), it follows that

$$\begin{aligned} p''(t) =& \eta''(t) - u''(t) = f(t, w, Tw) + M\eta(t) + N(T\eta)(t) - Mw(t) - N(Tw)(t) - \\ & f(t, z, Tz) - M\mu(t) - N(T\mu)(t) + Mz(t) + N(Tz)(t) \\ \geq & Mp(t) - N(Tp)(t) \end{aligned}$$

obviously, $p(0) = p(a), p'(0) = p'(a)$. In view of Lemma 2, it implies that $p(t) \leq 0$ on J, that is, $Bw \leq Bz$.

(III) Defining the sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ with $u_n = Bu_n, v_n = Bv_n (n = 1, 2, \cdots)$. From (II) just proved, we get

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq u_n \leq \cdots \leq u_1 \leq u_0. \tag{12}$$

On account of the definition of u_n and (9), (10), we have

$$u_n(t) = \frac{e^{-Lt}}{e^{La} - 1} \int_0^a e^{Ls}[\overline{A}u_n(s) + Lu_n(s)]\mathrm{d}s + e^{-Lt} \int_0^t e^{Ls}[\overline{A}u_n(s) + Lu_n(s)]\mathrm{d}s \tag{13}$$

$$\begin{aligned} u_n'(t) = (\overline{A}u_n)(t) =& \frac{e^{-Lt}}{e^{La} - 1} \int_0^a e^{Ls}[Lu_n'(s) + Mu_n(s) + N(Tu_n)(s) + z_{n-1}(s)]\mathrm{d}s + \\ & e^{-Lt} \int_0^t e^{Ls}[Lu_n'(s) + Mu_n(s) + N(Tu_n)(s) + z_{n-1}(s)]\mathrm{d}s, \end{aligned} \tag{14}$$

where

$$z_{n-1}(t) = f(t, u_{n-1}(t), (Tu_{n-1})(t)) - Mu_{n-1}(t) - N(Tu_{n-1})(t). \tag{15}$$

Similar to the proof of (I), we can get

$$\begin{aligned} \|u_{n+i} - u_n\| \leq & b_1\|u_{n+i} - u_n\|_1 + \\ & \max_J\{\frac{1}{e^{La} - 1}| \int_0^a e^{Ls}[\frac{e^{-La}}{e^{La} - 1} \int_0^a e^{L\tau}(z_{n+i-1}(\tau) - z_{n-1}(\tau))\mathrm{d}\tau + \\ & e^{-Ls} \int_0^s e^{L\tau}(z_{n+i-1}(\tau) - z_{n-1}(\tau))\mathrm{d}\tau]\mathrm{d}s| + \\ & | \int_0^t e^{Ls}[\frac{e^{-Ls}}{e^{Ls} - 1} \int_0^a e^{L\tau}(z_{n+i-1}(\tau) - z_{n-1}(\tau))\mathrm{d}\tau + e^{-Ls} \int_0^s e^{L\tau}(z_{n+i-1}(\tau) - z_{n-1}(\tau))\mathrm{d}\tau]\mathrm{d}s|\} \\ \leq & b_1\|u_{n+i} - u_n\|_1 + \frac{2e^{La}a^2}{(e^{La} - 1)^2}\|z_{n+i-1} - z_{n-1}\|, \end{aligned}$$

$$\begin{aligned} \|u_{n+i}' - u_n'\| \leq & b_2\|u_{n+1} - u_n\|_1 + \\ & \max_J\{\frac{1}{e^{La} - 1}| \int_0^a e^{Ls}[z_{n+i-1}(s) - z_{n-1}(s)]\mathrm{d}s| + \int_0^t e^{Ls}[z_{n+i-1}(s) - z_n(s)]\mathrm{d}s|\} \\ \leq & b_2\|u_{n+i} - u_n\|_1 + \frac{ae^{La}}{e^{-La} - 1}\|z_{n+i-1} - z_{n-1}\|. \end{aligned}$$

Let $b^* = \max\{b_1, b_2\}, a^* = \max\{\frac{2a^2e^{La}}{(e^{La}-1)^2}, \frac{ae^{La}}{e^{La}-1}\}$, then the assumption (H_3) implies $b^* < 1$. Therefore, from the above two inequalities it follows that

$$\|u_{n+1} - u_n\| \leq \frac{a^*}{1-b^*}\|z_{n+i-1} - z_{n-1}\| \quad (n, i = 1, 2, \cdots). \tag{16}$$

Since the regularity of P leads to the normality of P, we see from (12) that $\{u_n\}_{n=1}^{\infty}$ is a bounded set in $C(J, E)$. Let $B_r = \{x \in E : |x| \leq r\}, f \in C(J \times E \times E, E)$ implies that f is bounded on $J \times B_r \times B_r$. So by (15), there exists a constant $c > 0$ such that $\|z_{n-1}\| \leq c(n = 1, 2, \cdots)$.

From (13), (14) and (15), it follows that $\{u_n\}_{n=1}^{\infty}$ is equicontinuous on J. On the other hand, in view of the regularity of P and (12), we get that $u_n(t) \to x(t)$ for $t \in J, n \to \infty$ and some $x \in C(J, E)$. Applying the Acoli-Arzela theorem, we obtain that $\{u_n\}_{n=1}^{\infty}$ converge uniformly and monotonically to $x(t)$ on J, that is,

$$\|u_n - x\| \to 0 \quad (n \to \infty). \tag{17}$$

From (15) and (17), we find

$$\|z_{n-1} - z\| \to 0 \quad (n \to \infty), \tag{18}$$

where

$$z(t) = f(t, x(t), Tx(t)) - Mx(t) - N(Tx)(t). \tag{19}$$

Now, from (16),(17), it follows that $\{u_n\}_{n=1}^{\infty}$ is convergent on $C^1(J, E)$, and hence, (17) implies that $x \in C^1(J, E)$ and

$$\|u_n - x\|_1 \to 0 \quad (n \to \infty). \tag{20}$$

Observing (18), (19) and (20) and taking limits in (13) and (14), respectively, we have

$$\begin{aligned}
x(t) =& \frac{e^{-Lt}}{e^{La}-1}\int_0^a e^{Ls}[Lx(s) + x'(s)]\mathrm{d}s + e^{-Lt}\int_0^t e^{Ls}[Lx(s) + x'(s)]\mathrm{d}s, \\
x'(t) =& (\overline{A}x)(t) = \frac{e^{-Lt}}{e^{La}-1}\int_0^a e^{Ls}[Lx'(s) + f(s, x(s), (Tx)(s))]\mathrm{d}s + \\
& e^{-Lt}\int_0^t e^{Ls}[Lx'(s) + f(s, x(s), (Tx)(s))]\mathrm{d}s.
\end{aligned}$$

This implies by virtue of Lemma 3 that x is a solution of PBVP(1).

In the same way, we can find $y \in C^1(J, E)$ such that $\|v_n - y\|_1 \to 0(n \to \infty)$, and y is a solution of PBVP(1).

(IV) Finally, we show that x, y are minimal and maximal solutions in $[v_0, u_0]$ for PBVP(1). To this end, assume that $u \in [v_0, u_0]$ is any solution of PBVP(1) and $v_{n-1}(t) \leq u(t) \leq u_{n-1}(t)$ for $t \in J$, let $p(t) = u(t) - u_n(t)$, by (H_2)(ii) we get

$$\begin{aligned}
p'' &= u'' - u_n'' = f(t, u, Tu) - Mu_n - N(Tu_n) - f(t, u_{n-1}, Tu_{n-1}) + Mu_{n-1} + N(Tu_{n-1}) \\
&\geq M(u - u_{n-1}) + N(Tu - Tu_{n-1}) + M(u_{n-1} - u_n) + N(Tu_{n-1} - Tu_n) \\
&= M((u - u_n) + N(Tu - Tu_n) = Mp + NTp,
\end{aligned}$$

and $p(0) = p(a), p'(0) = p'(a)$. Then $p(t) \leq 0$ on J because of Lemma 2, i.e., $u \leq u_n$. In the same way, we can show that $v_n \leq u$. Hence, by induction, we have that $v_n \leq u \leq u_n$ for $n = 1, 2, \cdots$. Let n go to infinity, we have $y \leq u \leq x$. The proof of Theorem 2 is completed.

Remark The condition that P is regular can be omitted if E is weakly sequentially complete and P is normal.

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Banach 空间中二阶积分微分方程的周期边值问题

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摘 要: 通过建立对比结果，用上解和下解的方法，本文获得了二阶积分微分方程的周期边值问题最大最小解的存在性定理.