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The Robust Stability for Interval-Polynomial *

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Abstract: In this paper, a new stability criterion for positive-coefficient polynomial is given. Then the problem about the robust stability for an interval-polynomial is investigated and some new stability criterions for interval-polynomials are obtained. The coefficient perturbation bound for stable interval polynomial can be completely determined by the coefficients of polynomial (1.1). So the conclusions of this paper are simple and useful. Several examples in the end of this paper show that the criterions given in this paper are effective.

Key words: polynomial stability; interval polynomial; Robust stability; perturbed coefficients.

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1. Introducation and results

The real polynomial

$$f(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n \tag{1.1}$$

is called stable if every one of its roots has negative real part, where $a_i > 0$ $(i = 1, 2, \cdots, n)$. If coefficients a_i of polynomial (1.1) vary in $[\underline{a_i}, \overline{a_i}]$, then it is called an internal polynomial which is written as the following form

$$\widetilde{f}(s) = \widetilde{a_0} s^n + \widetilde{a_1} s^{n-1} + \cdots + \widetilde{a_n}, \tag{1.2}$$

where $0 < \underline{a_i} < \widetilde{a_i} < \overline{a_i}$. Interval polynomial (1.2) is called robust stable if it is stable for any $0 < \underline{a_i} < \widetilde{a_i} < \overline{a_i}$.

It is well know that the problem of robust stability for linear time-invariant uncertain system is related to the stability for a interval polynomial. So researching the stability for interval-polynomial draws a considerable interest. And many results [1-4] have been obtained in recent years.

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In this paper, we investigate this problem again, obtain some new sufficient conditions for the robust stability of an interval polynomial . The criterions given in this paper are easier to carry out than the others.

2. Main results

We know from [5] that the polynomial (1.1) is stable if and only if the $(n-1)$th degree polynomial

$$F(s)=a_0a_1s^{n-1}+(a_1a_2-a_0a_3)s^{n-2}+\cdots+a_0a_{2k-1}s^{n-(2k-1)}+ (a_1a_{2k}-a_0a_{2k+1})s^{n-2k}+\cdots+\begin{cases} a_0a_n & (n \text{ is odd}) \\ a_1a_n & (n \text{ is even}) \end{cases}$$

is stable. Let

$$a_{0,0}=a_0a_1, a_{0,1}=a_1a_2-a_0a_3,\cdots, a_{0k}=a_1a_{2k}-a_0a_{2k+1},$$

$$a_{0,n-1}=\begin{cases} a_0a_n & (n \text{ is odd}) \\ a_1a_n & (n \text{ is even}) \end{cases}$$

$$a_{i,0}=a_{i-1,0}a_{i-1,1}, a_{i,1}=a_{i-1,1}a_{i-1,2}-a_{i-1,0}a_{i-1,3},\ \cdots,\ a_{i,k}=a_{i-1,1}a_{i-1,2k}-a_{i-1,0}a_{i-1,2k+1},$$

$$a_{i,m-1}=\begin{cases} a_{i-1,0}a_{i-1,m} & (m \text{ is odd}) \\ a_{i-1,1}a_{i-1,m} & (m \text{ is even}) \end{cases} \quad (i=1,2,\cdots,n-2)$$

And let

$$\alpha_{0,i}=\frac{a_{i-1}a_{i+2}}{a_ia_{i+1}} \quad (i=1,2,\cdots,n-2),$$

$$\alpha_{i,2k-1}=\frac{\alpha_{i-1,2k}(1-\alpha_{i-1,2k+1}!!)}{1-\alpha_{i-1,2k-1}!!},\ \alpha_{i,2k}=\frac{\alpha_{i-1,2k+1}(1-\alpha_{i-1,2k-1}!!)}{1-\alpha_{i-1,2k+1}!!} \quad (i=1,2,\cdots,n-3).$$

Where $\alpha_{i,2k-1}!!=\alpha_{i,1}\alpha_{i,3}\cdots\alpha_{i,2k-1}$, $\alpha_{i,2k}!!=\alpha_{i,2}\alpha_{i,4}\cdots\alpha_{i,2k}$. Then we have the following conclusion.

Lemma 2.1 *A necessary and sufficient condition for the stability of positive coefficient polynomial (1.1) is that:*

$$\alpha_{j,1}<1, \alpha_{j,1}\alpha_{j,3}<1,\cdots,\alpha_{j,1}\alpha_{j,3}\cdots\alpha_{j,2k-1}<1,\ 1\le 2k-1\le n-(j-2),\ j=0,1,\cdots,n-3.$$

The proof of this Lemma is omitted.

Clearly for $n=3$, the polynomial is stable if and only if $\alpha_{0,1}<1$; $n=4$, the polynomial is stable if and only if $\alpha_{0,1}<1,\ \alpha_{1,1}<1$.

Using Lemma 2.1, we can deduce the following conclusion.

Theorem 2.2 *If for* $n=3, \overline{\alpha}_{0,1}<1$; *for* $n=4, \overline{\alpha}_{0,i}<\frac{1}{2}$ $(i=1,2)$; *And* $n\ge 5$,

$$\overline{\alpha}_{0,j}<\frac{1}{2}(j=1,n-2), \overline{\alpha}_{0,j}<\frac{1}{4}(j=2,\cdots,n-3),$$

where $\overline{\alpha}_{0,i} = \frac{\overline{a}_{i-1}\overline{a}_{i+2}}{\underline{a}_i\underline{a}_{i+1}} (i = 1, 2, \cdots, n-2)$. *Then the internal polynomial (1.2) is stable for* $\tilde{a}_i \in (\underline{a}_i, \overline{a}_i)$.

Proof Clearly Theorem 2.2 is true when n=3. For n=4, we know that

$$\alpha_{0,1} < \overline{\alpha}_{0,1}, \alpha_{1,1} = \frac{\alpha_{0,2}(1-\alpha_{0,1}\alpha_{0,3})}{1-\alpha_{0,1}} < \frac{\overline{\alpha}_{0,2}}{1-\overline{\alpha}_{0,1}} = \overline{\alpha}_{1,1},$$

so that when $\overline{\alpha}_{0,i} < \frac{1}{2}$ $(i = 1, 2)$, we have $\overline{\alpha}_{0,1} < 1$, $\overline{\alpha}_{1,1} < 1$. That is to say the interval polynomial (1.2) is stable. So Theorem 3.1 is true when n=4. For $n = 5$, we have

$$\alpha_{2,1} < \frac{\overline{\alpha}_{1,2}}{1-\overline{\alpha}_{1,1}}, \overline{\alpha}_{1,1} = \frac{\overline{\alpha}_{0,2}}{1-\overline{\alpha}_{0,1}}, \alpha_{1,1} = \frac{\alpha_{0,2}(1-\alpha_{0,1}\alpha_{0,3})}{1-\alpha_{0,1}} < \overline{\alpha}_{1,1}.$$

Because $\alpha_{0,i} < \overline{\alpha}_{0,i}$ $(i = 1, 2, 3)$, we know that $\overline{\alpha}_{0,1} < 1$, $\overline{\alpha}_{0,1}\overline{\alpha}_{0,3} < 1, \overline{\alpha}_{1,1} < 1$, $\overline{\alpha}_{2,1} < 1$, if $\overline{\alpha}_{0,j} < \frac{1}{2}(j = 1, 3), \overline{\alpha}_{0,2} < \frac{1}{4}$. In general, let Theorem 2.2 is true when $n = k (k \geq 5)$. When $n = k+1$, due to

$$\overline{\alpha}_{1,1} = \frac{\overline{\alpha}_{0,2}(1-\overline{\alpha}_{0,1}\overline{\alpha}_{0,2})}{1-\overline{\alpha}_{0,1}} < \frac{1/4}{2} = \frac{1}{8} < \frac{1}{2},$$

$$\overline{\alpha}_{1,2} = \frac{\overline{\alpha}_{0,3}(1-\overline{\alpha}_{0,1})}{1-\overline{\alpha}_{0,1}\overline{\alpha}_{0,3}} < \frac{1/4}{7/8} = \frac{7}{32} < \frac{1}{4},$$

$$\cdots, \cdots, \cdots, \cdots$$

$$\overline{\alpha}_{1,k-3} = \frac{\overline{\alpha}_{0,k-2}(1-\overline{\alpha}_{0,k-1}!!)}{1-\overline{\alpha}_{0,k-2}!!} < \frac{\overline{\alpha}_{0,k-2}}{1-\overline{\alpha}_{0,k-2}!!} < \frac{1/4}{4/3} = \frac{3}{16} < \frac{1}{4},$$

$$\overline{\alpha}_{1,k-2} = \frac{\overline{\alpha}_{0,k-1}(1-\overline{\alpha}_{0,k-3}!!)}{1-\overline{\alpha}_{0,k-1}!!} < \frac{\overline{\alpha}_{0,k-1}}{1-\overline{\alpha}_{0,k-1}!!} < \frac{1/2}{2} = \frac{1}{4} < \frac{1}{2},$$

so that if

$$\overline{\alpha}_{0,j} < \frac{1}{2}(j = 1, k-1), \overline{\alpha}_{0,j} < \frac{1}{4} \quad (j = 2, \cdots, k-2),$$

then kth degree polynomial

$$\overline{F}(s) = \overline{a}_0\overline{a}_1 s^k + (\overline{a}_1\overline{a}_2 - \overline{a}_0\overline{a}_3)s^{k-1} + \cdots + \overline{a}_0\overline{a}_{2k-1}s^{k-(2m-1)} + \\ (\overline{a}_1\overline{a}_{2k} - \overline{a}_0\overline{a}_{2k+1})s^{k-2m} + \cdots + \begin{cases} \overline{a}_0\overline{a}_m & (m \text{ is odd}) \\ \overline{a}_1\overline{a}_m & (m \text{ is even}) \end{cases}$$

is stable.By [5], we know the (k+1)th degree interval polynomial (1.2) is stable. By the supposition of inductive method, Theorem 2.2 holds for any n. We complete the proof.

Notice. For $n = 3, \overline{\alpha}_{0,1} < 1$ is necessary and sufficient condition for the stable interval polynomial.

Theorem 2.3 *If let x be perturbation term of coefficient of (1.2), i.e., $\tilde{a}_i = a_i + x$ then the interval polynomial (1.2) is stable under the following conditions:*

$$x < \min\{r_1, r_2\}, \quad \textit{for} \ \ n = 4,$$

$$x < \min\{r_1, r_2, \cdots, r_{n-2}\}, \quad \textit{for} \quad n \geq 5,$$

where

$$r_i = \frac{-2(a_{i-1} + a_{i+2}) - (a_i + a_{i+1}) + \sqrt{4(a_{i-1}^2 - a_{i-1}a_{i+2} + a_{i-1}^2) + (a_i + a_{i+1})^2 + 4\sum_{k=i-1}^{i+1}\sum_{j=k+1}^{i+2} a_k a_j}}{2}$$

$$(i = 1, n-2; n \geq 4),$$

$$r_j = \frac{-2(a_{j-1} + a_{j+2}) - (a_j + a_{j+1}) + \sqrt{16(a_{j-1}^2 - a_{j-1}a_{j+2}/2 + a_{j+1}^2) + (a_j + a_{j+1})^2 + 8\sum_{k=j-1}^{j+1}\sum_{m=k+1}^{j+2} a_k a_m}}{6}$$

$$(j = 2, 3, \cdots, n-3; n \geq 5).$$

Proof By Theorem 2.2, if

$$\overline{\alpha}_{0,1} = \frac{(a_0 + x)(a_3 + x)}{(a_1 - x)(a_2 - x)} < \frac{1}{2},$$

$$\overline{\alpha}_{0,j} = \frac{(a_{i-1} + x)(a_{i+2} + x)}{(a_i - x)(a_{i+1} - x)} < \frac{1}{4} \quad (i = 2, \cdots, n-3),$$

$$\overline{\alpha}_{0,n-2} = \frac{(a_{n-30} + x)(a_n + x)}{(a_{n-2} - x)(a_{n-1} - x)} < \frac{1}{2},$$

then interval polynomial (1.2) is stable. Solving above inequalities, we can get

$$x < \min\{r_1, r_2\}, \quad \text{for} \quad n = 4,$$

$$x < \min\{r_1, r_2, \cdots, r_{n-2}\}; \quad \text{for} \quad n \geq 5.$$

So the proof is over. In similar way, we can get another conclusion as follows:

Theorem 2.4 If

$$x < \min\{m_1, m_2\}, \quad \text{for} \quad n = 4,$$

$$x < \min\{m_1, m_{2,} \cdots m_{n-2}\}, \quad \text{for} \quad n \geq 5,$$

where $m_i = \frac{(1-\sqrt{2\alpha_{0,i}})^2}{1-2\alpha_{0,i}}$, $(i = 1, n-2, n \geq 4)$; $m_j = \frac{(1-\sqrt{4\alpha_{0,j}})^2}{1-4\alpha_{0,j}}$, $(j = 2, \cdots, n-3, n \geq 5)$. Then the interval polynomial (1.2) is stable, in which $\tilde{a}_i = a_i \pm a_i x, i = 0, 1, \cdots, n$.

3. Examples

Example 1 Consider a 4th degree interval polynomial:

$$\tilde{f}(s) = (1 \pm x)s^4 + (5 \pm x)s^3 + (8 \pm x)s^2 + (8 \pm x)s + (3 \pm x). \tag{3.1}$$

Clearly $\alpha_{0,1} = \frac{1}{5} < 1$, $\alpha_{1,1} = \frac{75}{256} < 1$, so that polynomial (3.1) is stable when x=0. And due to $r_1 \approx 1.029$, $r_2 \approx 0.76$. Hence if $x < 0.76$, then polynomial is stable.That is to say the polynomial

$$\tilde{f}(s) = (1 \pm 0.76)s^4 + (5 \pm 0.76)s^3 + (8 \pm 0.76)s^2 + (8 \pm 0.76)s + (3 \pm 0.76)$$

is stable.

Example 2 Given $g(s) = s^5 + 6s^4 + 8s^3 + 10s^2 + 3s + 1/2$. Clearly

$$\alpha_{0,1} \approx 0.208, \alpha_{0,2} \approx 0.225, \alpha_{0,3} \approx 0.133, \alpha_{1,1} \approx 0.276,$$

so g(s) is stable. And due to

$$m_1 \approx 0.129\ ,\ m_2 \approx 0.026,\ m_3 \approx 0.318,$$

we know the following polynomial is also stable.

$$\begin{aligned}\tilde{g}(s) =&(1 \pm 0.026)s^5 + (6 \pm 0.156)s^4 + (8 \pm 0.208)s^3 + (10 \pm 0.26)s^2 + \\ &(3 \pm 0.078)s + (1/2 \pm 0.013)\end{aligned}$$

4. Conclusion

In this paper, we discuss the problem about the robust stability for interval polynomial and obtain some sufficient conditions . Using the result given in this paper, we can easily get the perturbation bound of coefficient for stable interval polynomial, which is completely determined by the coefficients of polynomial (1.1).

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区间多项式的鲁棒稳定性

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摘 要: 本文首先给出了一种新的判定多项式稳定的充要条件 (引理 2.1). 然后，在此基础上，研究了区间多项式的鲁棒稳定性，得到了若干判别区间多项式的充分条件 (定理 2.2-定理 2.3). 由于所得的摄动界完全可由原末被扰动的多项式的系数所决定，这使得本文的方法比现有的结果简单好用. 文末的例子说明了本文方法的有效性.