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Exact Convergence Rates of Functional Modulus of Continuity of a Wiener Process *

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Abstract: Let $\{W(t), t \geq 0\}$ be a standard Wiener process and S be the set of Strassen's functions. In this paper we investigate the exact rates of convergence to zero of the variables $\sup_{0\leq t\leq 1-h}\inf_{f\in S}\sup_{0\leq x\leq 1}|(W(t+hx)-W(t))(2h\log h^{-1})^{-1/2}-f(x)|$ and $\inf_{0\leq t\leq 1-h}\sup_{0\leq x\leq 1}|(W(t+hx)-W(t))(2h\log h^{-1})^{-1/2}-f(x)|$ for any $f\in S$. As a consequence, a relation between the modulus of non-differentiability and the functional modulus of continuity for a Wiener process is established.

Key words: Wiener process; functional modulus of continuity; modulus of non-differentiability.

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1. Introduction and results

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $\{W(t), t \geq 0\}$ be a standard Wiener process defined on it. Denote by $C_0[0,1]$ the set of continuous functions on $[0,1]$ with value zero at the origin. Let $S \subset C_0[0,1]$ be the class of functions defined in Strassen's law of the iterated logarithm (cf. [1]), i.e., $f(x) \in S$ if and only if $f(0) = 0$, $f(x)$ is absolutely continuous and $\int_0^1 (f'(x))^2 dx \leq 1$. Define $\|f\|_\infty = \sup_{0\leq x\leq 1}|f(x)|$ for any $f \in C_0[0,1]$. Moreover, introduce the following notations:

$$\begin{aligned} &L(h) = \log h^{-1},\\ &\beta_h = (2hL(h))^{-1/2},\\ &Y_{t,h}(x) = \beta_h(W(t+hx) - W(t)),\quad 0\leq x\leq 1,\quad 0<h<1,\quad 0\leq t\leq 1-h, \end{aligned} \tag{1.1}$$

where log is the natural logarithm.

The following result is the well-known modulus of continuity theorem of a Wiener process (cf. Theorem 1.1.1 in [2]):

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Theorem A *We have*

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\sup_{0<s\le h}\frac{|W(t+s)-W(t)|}{\sqrt{2h\log h^{-1}}}=\lim_{h\to 0}\sup_{0\le t\le 1-h}\frac{|W(t+h)-W(t)|}{\sqrt{2h\log h^{-1}}}=1 \quad \text{a.s.} \tag{1.2}$$

Mueller[3] and Chen[4] combined Strassen's idea with Lévy's and established functional modulus of continuity for a wiener process saparately, which also imply Theorem A. They obtained

Theorem B *We have*

$$\lim_{h\to 0}\sup_{0\le t\le 1-h}\inf_{f\in S}\|Y_{t,h}-f\|_\infty=0 \quad \text{a.s.}, \tag{1.3}$$

and for each $f\in S$,

$$\lim_{h\to 0}\inf_{0\le t\le 1-h}\|Y_{t,h}-f\|_\infty=0 \quad \text{a.s.} \tag{1.4}$$

The meaning of this theorem let us mention that:

(a) For all h *small enough and for every* $0\le t\le 1-h$ *the function* $Y_{t,h}(x)$ *can be approximated by a suitable element* $f(x)\in S$ *uniformly on* $[0,1]$.

(b) For all h *small enough and for any* $f(x)\in S$ *there exists* a $0<t<1$ *such that* $Y_{t,h}(x)$ *will approximate the given* $f(x)$ *uniformly on* $[0,1]$.

The aim of the present paper is to investigate the exact rates of convergence of (1.3) and (1.4). Consequently, however, our results are inspired by the discussions in [5-8], where they studied the rates of convergence of Strassen's law of the iterated logarithm of a Wiener process. As a consequence, we establish a relation between the modulus of non-differentiability and functional modulus of continuity for a Wiener process.

For use later on, define

$$I(f)=\begin{cases}\displaystyle\int_0^1 (f'(x))^2\mathrm{d}x & \text{if } f \text{ is an absolutely continuous function,}\\ \infty & \text{otherwise,}\end{cases}$$

for any $f\in C_0[0,1]$.

The following is our results:

Theorem 1.1 *There exists a constant* $0<\gamma<\infty$, *which is independent of h, such that*

$$P\Big(\sup_{0\le t\le 1-h}\inf_{f\in S}\|Y_{t,h}-f\|_\infty\ge\gamma\Big(\frac{\log L(h)}{L(h)}\Big)^{2/3}\quad \text{i.o.}\Big)=0. \tag{1.5}$$

Theorem 1.2 *For any* $f\in S$,

$$\lim_{h\to 0}\inf_{0\le t\le 1-h}\|Y_{t,h}-f\|_\infty L(h)=\begin{cases}\dfrac{\pi}{4\sqrt{1-I(f)}} & \text{if } I(f)<1,\\ \infty & \text{if } I(f)=1,\end{cases}\quad \text{a.s.} \tag{1.6}$$

If $f(x)\equiv 0$ in (1.6), then we have the following

Corollary 1.1 *We have*

$$\lim_{h\to 0}\inf_{0\le t\le 1-h}\sup_{0\le s\le h}\left(\frac{L(h)}{2h}\right)^{1/2}|W(t+s)-W(t)|=\frac{\pi}{4}\quad \text{a.s.} \tag{1.7}$$

Remark 1.1 (1.7) is the well-known modulus of non-differentiability of a Wiener process proved by Csörgő and Révész (cf. [2]).

The discussion of the case $I(f)=1$ seems to be more difficult, we can give the best rate only if $f(x)$ is piecewise linear. Let $f(x)$ be a continuous broken line with $f(0)=0$, and

$$f'(x)=\beta_i,\quad a_{i-1}<x<a_i\quad (i=1,2,\dots,k), \tag{1.8}$$

where $a_0=0<a_1<a_2<\dots<a_k=1$.

Theorem 1.3 *If $f(x)$ is defined as above and $I(f)=1$, then there exist two constants c_1 and c_2 such that*

$$c_1\le \lim_{h\to 0}\inf_{0\le t\le 1-h}\|Y_{t,h}-f\|_\infty (L(h))^{2/3}\le c_2\quad \text{a.s.}, \tag{1.9}$$

where $c_1<\pi^{2/3}2^{-5/3}B^{-1/3}$ and $c_2>\pi^{2/3}2^{-5/3}B^{-1/3}$. Here

$$B=|\beta_2-\beta_1|+\dots+|\beta_k-\beta_{k-1}|+|\beta_k|.$$

2. Proofs of the theorems.

The proofs are based on the following two lemmas.

Lemma 2.1 *Let $\varepsilon_\lambda:=C_0(\lambda^{-2}\log\lambda)^{2/3}$ $(\lambda>0)$, where $C_0>0$ is a constant. Then there exist two constants $\lambda_0>0$ and $h_0>0$ such that for any $\lambda\ge\lambda_0$ and $h\le h_0$,*

$$P\Big(\sup_{0\le t\le 1-h}\inf_{f\in S}\Big\|\frac{W(t+h\cdot)-W(t)}{\sqrt{h}}-\lambda f\Big\|_\infty\ge\lambda\varepsilon_\lambda\Big)\le\frac{C}{h}\exp\Big(-\frac{\lambda^2(1+\varepsilon_\lambda)^2}{2}+C\varepsilon_\lambda^{-1/2}\log\lambda\Big).$$

Here, and in the sequel, C stands for a positive constant whose value is uninteresting and may vary for each appearance.

Proof See Lemma 2.4 in [9]. □

Lemma 2.2 *For any $\delta>0$ and $f\in S$,*

$$\lim_{\lambda\to\infty}\lambda^{-2}\log P(\|W-\lambda f\|_\infty\le\lambda^{-1}\delta)=-\frac{\pi^2}{8}\delta^{-2}-\frac{1}{2}I(f).$$

Proof See Theorem 3.3 in [5]. □

Now we are ready to prove the theorems.

Proof of Theorem 1.1 Define

$$h_n=e^{-n^{\frac{1}{3}}},\ n\ge 1. \tag{2.1}$$

For any $h \in (0,1)$, there exists a positive integer n such that $h_{n+1} \le h \le h_n$. Then we have

$$\begin{aligned}
&\sup_{0\le t\le 1-h}\inf_{f\in S}\|Y_{t,h}-f\|_\infty \\
&\le \sup_{0\le t\le 1-h_{n+1}}\inf_{f\in S}\|Y_{t,h_{n+1}}-f\|_\infty + \sup_{0\le t\le 1-h_{n+1}}\sup_{0\le x\le 1}|Y_{t,h_{n+1}}(\frac{h}{h_{n+1}}x)-Y_{t,h_{n+1}}(x)|+ \\
&\sup_{0\le t\le 1-h_{n+1}}\sup_{0\le x\le 1}(1-\frac{\beta_{h_n}}{\beta_h})|Y_{t,h_n}(x)| + 2\sup_{f\in S}\sup_{0\le x\le 1}|f(\frac{h}{h_{n+1}}x)-f(x)| \\
&=: I_1^{(n)}+I_2^{(n)}+I_3^{(n)}+I_4^{(n)}. \qquad (2.2)
\end{aligned}$$

By Lemma 2.1 (with $\lambda = \sqrt{2L(h_{n+1})}$), we have

$$P(\sup_{0\le t\le 1-h_{n+1}}\inf_{f\in S}\|Y_{t,h_{n+1}}-f\|_\infty \ge \gamma(\frac{\log L(h_{n+1})}{L(h_{n+1})})^{2/3}) \le \frac{C}{(n+1)^2} \qquad (2.3)$$

by taking $\gamma > 0$ large enough. Hence by the Borel-Cantelli lemma we get

$$P(I_1^{(n)} \ge \gamma(\frac{\log L(h_{n+1})}{L(h_{n+1})})^{2/3} \quad \text{i.o.}) = 0. \qquad (2.4)$$

By the difinition of h_n, it is easy to see that $h_n = \exp(-\frac{n}{(L(h_n))^2}), n \ge 1$. Then we have

$$1 \ge \frac{h_{n+1}}{h_n} \ge \exp(-\frac{1}{(L(h_n))^2}) \to 1 \quad \text{as } n\to\infty. \qquad (2.5)$$

Therefore

$$1-(\frac{h_{n+1}}{h_n})^C \le \frac{C}{(L(h_n))^2}. \qquad (2.6)$$

To consider $I_2^{(n)}$ and $I_3^{(n)}$. By Theorem 1.2.1 in [2], for sufficiently large n and for any $0<\delta<1/4$ we have (with probability one when necessary)

$$\begin{aligned}
I_2^{(n)} &\le \sup_{0\le\tau\le 2}\sup_{0\le s\le h_n-h_{n+1}}\frac{|W(\tau+s)-W(\tau)|}{\sqrt{2(h_n-h_{n+1})\log(2(h_n-h_{n+1})^{-1})}}\times \\
&\quad \frac{\sqrt{2(h_n-h_{n+1})\log(2(h_n-h_{n+1})^{-1})}}{\sqrt{2h_{n+1}\log h_{n+1}^{-1}}} \\
&\le 2\frac{\sqrt{(h_n-h_{n+1})\log(2(h_n-h_{n+1})^{-1})}}{\sqrt{h_{n+1}\log h_{n+1}^{-1}}} \le 2(\frac{h_n}{h_{n+1}})^{1/2+\delta}(1-\frac{h_{n+1}}{h_n})^{1/2+\delta} \\
&\le \frac{4}{(L(h_n))^{1+2\delta}} \quad \text{a.s.} \qquad (2.7)
\end{aligned}$$

Similarly,

$$I_3^{(n)} \le 2(1-\frac{\beta_{h_n}}{\beta_{h_{n+1}}}) \le 2(1-(h_{n+1}/h_n)^{1/2}) \le 1/(L(h_n))^2 \quad \text{a.s.} \qquad (2.8)$$

Finally we consider $I_4^{(n)}$. Since $f\in S$, we have

$$I_4^{(n)} \le 2\sup_{f\in S}(I(f))^{1/2}(\frac{h_n}{h_{n+1}}-1)^{1/2} \le 4(1-\frac{h_{n+1}}{h_n})^{1/2} \le \frac{4}{L(h_n)}. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we get

$$P(I_2^{(n)}+I_3^{(n)}+I_4^{(n)} \ge \gamma(\frac{\log L(h_{n+1})}{L(h_{n+1})})^{2/3} \quad \text{i.o.}) = 0. \tag{2.10}$$

Therefore by (2.2), (2.4) and (2.10), we obtain (1.5) and complete the proof of Theorem 1.1. □

Proof of Theorem 1.2 For $f\in S$, put $\kappa(f) := \frac{\pi}{4\sqrt{1-I(f)}}$. For proving (1.6), it is enough to show that for $I(f)<1$ we have

$$\liminf_{h\to 0}\inf_{0\le t\le 1-h} L(h)\|Y_{t,h}-f\|_\infty \ge \kappa(f) \quad \text{a.s.}, \tag{2.11}$$

$$\limsup_{h\to 0}\inf_{0\le t\le 1-h} L(h)\|Y_{t,h}-f\|_\infty \le \kappa(f) \quad \text{a.s.}, \tag{2.12}$$

and that for $I(f)=1$ we have

$$\liminf_{h\to 0}\inf_{0\le t\le 1-h} L(h)\|Y_{t,h}-f\|_\infty \ge M \quad \text{a.s.} \tag{2.13}$$

for any $M>0$.

At first we show (2.11). Define $h_n := e^{-n^{1/3}}$, $n\ge 1$. For $\omega\in\Omega$, define $Z(h,\omega) := \inf_{0\le t\le 1-h} L(h)\|Y_{t,h}-f\|_\infty$ and $Z_n^{(1)}(\omega) := \inf_{h_{n+1}\le h\le h_n} Z(h,\omega)$. For any $0<\varepsilon<1$ and $\omega\in\Omega$, by the definition of infimum, there exists $\tau_n=\tau_n(\omega)\in[h_{n+1},h_n]$ such that $Z_n^{(1)}(\omega)\ge Z(\tau_n,\omega)-\varepsilon$.

For convenient, we will not write the argument ω. Let u $(0\le u\le 1)$ be arbitrary and put $x=\frac{uh_{n+1}}{\tau_n}$. Then $0\le x\le \frac{h_{n+1}}{\tau_n}\le 1$. We can write

$$\begin{aligned}
&\inf_{0\le t\le 1-h_{n+1}}\sup_{0\le u\le 1}|W(t+uh_{n+1})-W(t)-f(u)\beta_{h_{n+1}}^{-1}| \\
&\le \inf_{0\le t\le 1-\tau_n}\sup_{0\le x\le 1}|W(t+x\tau_n)-W(t)-f(\frac{x\tau_n}{h_{n+1}})\beta_{h_{n+1}}^{-1}| \\
&\le \inf_{0\le t\le 1-\tau_n}\sup_{0\le x\le 1}\{|W(t+x\tau_n)-W(t)-f(x)\beta_{\tau_n}^{-1}|+ \\
&\quad |f(x)|(\beta_{\tau_n}^{-1}-\beta_{h_{n+1}}^{-1})+|f(x)-f(\frac{x\tau_n}{h_{n+1}})|\beta_{h_{n+1}}^{-1}\} \\
&\le \beta_{h_n}^{-1}L^{-1}(h_{n+1})Z(\tau_n)+(\beta_{h_n}^{-1}-\beta_{h_{n+1}}^{-1})+\sqrt{2(h_n-h_{n+1})L(h_{n+1})}.
\end{aligned} \tag{2.14}$$

Hence, by (2.14) we have

$$\liminf_{h\to 0} Z(h) \geq \liminf_{n\to\infty} Z_n^{(1)} \geq \liminf_{n\to\infty} Z(\tau_n) - \varepsilon$$
$$\geq \liminf_{n\to\infty} \beta_{h_n} L(h_{n+1}) \inf_{0\le t\le 1-h_{n+1}} \sup_{0\le u\le 1} |W(t+uh_{n+1}) - W(t) - f(u)\beta_{h_{n+1}}^{-1}| -$$
$$\limsup_{n\to\infty} \beta_{h_n} L(h_{n+1})\{(\beta_{h_n}^{-1} - \beta_{h_{n+1}}^{-1}) + \sqrt{2(h_n - h_{n+1})L(h_{n+1})}\} - \varepsilon$$
$$=: J_1 + J_2 - \varepsilon. \tag{2.15}$$

From (2.5) and (2.6), it is easy to see that

$$J_2 = 0. \tag{2.16}$$

Since $0 < \varepsilon < 1$ is arbitrary, for proving (2.11), it is enough to show that

$$J_1 \geq \kappa(f) \quad \text{a.s.} \tag{2.17}$$

We define

$$t_i = t_i^{(n)} = ih_{n+1}(L(h_{n+1}))^{-3} \quad i = 0,1,2,\cdots,\rho_{h_{n+1}} = [(h_{n+1})^{-1}(L(h_{n+1}))^3],$$

where $[x]$ denotes the integer part of x.

For proving (2.17), we first show that

$$\liminf_{n\to\infty} \min_{0\le i\le \rho_{h_{n+1}}} \sup_{0\le x\le 1} |W(t_i + h_{n+1}x) - W(t_i) - f(x)\beta_{h_{n+1}}^{-1}|\beta_{h_n} L(h_n) \geq \kappa(f) \quad \text{a.s.} \tag{2.18}$$

By Lemma 2.2, $\forall 0 < \varepsilon < 1, \forall 0 < \delta < 1$ and sufficiently large n we have

$$P(\min_{0\le i\le \rho_{h_{n+1}}} \sup_{0\le x\le 1} |W(t_i + h_{n+1}x) - W(t_i) - f(x)\beta_{h_{n+1}}^{-1}| < (1-\varepsilon)\beta_{h_n}^{-1} L^{-1}(h_n)\kappa(f))$$
$$\leq (\rho_{h_{n+1}} + 1) P(\|W - f\sqrt{2L(h_{n+1})}\|_\infty < \frac{(1-\varepsilon)\sqrt{2}\kappa(f)}{\sqrt{h_{n+1}}})$$
$$\leq (\rho_{h_{n+1}} + 1) \exp(-I(f)L(h_{n+1}) - \frac{1 - I(f)}{(1-\varepsilon)^2} L(h_{n+1}) + \delta L(h_{n+1}))$$
$$\leq C h_n^{\frac{1}{(1-\varepsilon)^2} - (\frac{1}{(1-\varepsilon)^2} - 1)I(f) - 1 - \delta} (L(h_n))^3.$$

Since $I(f) < 1$, we can take $\varepsilon > 0$ and $\delta > 0$ small enough such that $\frac{1}{(1-\varepsilon)^2} - (\frac{1}{(1-\varepsilon)^2} - 1)I(f) - 1 - \delta > 0$. Then, via the Borel-Cantelli lemma we get (2.18).

Note that

$$J_1 \geq \liminf_{n\to\infty} \beta_{h_n} L(h_n) \min_{0\le i\le \rho_{h_{n+1}}} \sup_{0\le x\le 1} |W(t_i + h_{n+1}x) - W(t_i) - f(x)\beta_{h_{n+1}}^{-1}| -$$
$$2\limsup_{n\to\infty} \beta_{h_n} L(h_n) \max_{0\le i\le \rho_{h_{n+1}}} \sup_{t_i\le t\le t_{i+1}} \sup_{0\le x\le 1} |W(t_i + h_{n+1}x) - W(t + h_{n+1}x)|$$
$$=: J_1' - J_1''. \tag{2.19}$$

By the modulus of continuity theorem (cf. Theorem A) we have

$$J_1'' \le 2\limsup_{n\to\infty}\beta_{h_n}L(h_n)\sup_{0\le\tau\le 1}\sup_{0\le s\le\rho_{h_{n+1}}^{-1}}\frac{|W(\tau+s)-W(\tau)|}{\sqrt{2\rho_{h_{n+1}}^{-1}\log\rho_{h_{n+1}}}}\cdot\sqrt{2\rho_{h_{n+1}}^{-1}\log\rho_{h_{n+1}}}$$

$$\le 4\limsup_{n\to\infty}L(h_n)\sqrt{\frac{h_{n+1}(L(h_{n+1}))^{-2}}{h_nL(h_n)}} = 0 \quad \text{a.s.} \tag{2.20}$$

Combining (2.18), (2.19) and (2.20), we obtain (2.17). This implies that (2.11) is proved.

Next we show (2.12). Let h_n be defined as in (2.1). Define

$$t_i = ih_n, i = 0,1,2,\ldots,p_{h_n} = [h_n^{-1}].$$

We first show that

$$\limsup_{n\to\infty}\min_{0\le i\le p_{h_n}}\|Y_{t_i,h_n}-f\|_\infty L(h_n)\le\kappa(f) \quad \text{a.s.} \tag{2.21}$$

By Lemma 2.2 (with $\lambda=\sqrt{2L(h_n)}, \delta=\frac{(1+\varepsilon)\sqrt{2}\kappa(f)}{\sqrt{L(h_n)}}$), $\forall 0<\varepsilon<1, \forall 0<\delta<1$, we have

$$\begin{aligned}
&P(\min_{0\le i\le p_{h_n}}\|Y_{t_i,h_n}-f\|_\infty L(h_n)\ge(1+\varepsilon)\kappa(f))\\
&\le (P(\|W-f\sqrt{2L(h_n)}\|_\infty > \frac{(1+\varepsilon)\sqrt{2}\kappa(f)}{\sqrt{L(h_n)}}))^{p_{h_n}+1}\\
&\le (1-\exp(-I(f)L(h_n)-\frac{1-I(f)}{(1+\varepsilon)^2}L(h_n)-\delta L(h_n)))^{p_{h_n}+1}\\
&\le \exp(-Ch_n^{\frac{1}{(1+\varepsilon)^2}-(\frac{1}{(1+\varepsilon)^2}-1)I(f)-1+\delta}(L(h_n))^3).
\end{aligned}$$

Since $I(f)<1$, we can take $\varepsilon>0$ and $\delta>0$ such that $\frac{1}{(1+\varepsilon)^2}-(\frac{1}{(1+\varepsilon)^2}-1)I(f)-1+\delta<0$. Then, via the Borel-Cantelli lemma we obtain (2.21). Note that for sufficiently large n,

$$\begin{aligned}
&\sup_{h_{n+1}\le h\le h_n}\inf_{0\le t\le 1-h}\|Y_{t,h}-f\|_\infty L(h)\\
&\le \sup_{h_{n+1}\le h\le h_n}\inf_{0\le t\le 1-h}\sup_{0\le s\le h}\beta_{h_{n+1}}L(h_{n+1})|W(t+s)-W(t)-f(s/h)\beta_h^{-1}|\\
&\le \sup_{h_{n+1}\le h\le h_n}\inf_{0\le t\le 1-h_n}\sup_{0\le s\le h_n}\beta_{h_{n+1}}L(h_{n+1})|W(t+s)-W(t)-f(s/h_n)\beta_{h_n}^{-1}|+\\
&\quad \beta_{h_{n+1}}L(h_{n+1})(\beta_{h_n}^{-1}-\beta_{h_{n+1}}^{-1})\sup_{0\le x\le 1}|f(x)|+\beta_{h_{n+1}}L(h_{n+1})\beta_{h_n}^{-1}\sup_{0\le s\le h_n}|f(\frac{s}{h_n})-f(\frac{s}{h_{n+1}})|\\
&\le \min_{0\le i\le p_{h_n}}\|Y_{t_i,h_n}-f\|_\infty L(h_n)+\beta_{h_{n+1}}L(h_{n+1})(\beta_{h_n}^{-1}-\beta_{h_{n+1}}^{-1})+2\beta_{h_{n+1}}L(h_{n+1})(1-\frac{h_{n+1}}{h_n})^{1/2}\\
&=: K_1^{(n)}+K_2^{(n)}+K_3^{(n)}.
\end{aligned} \tag{2.22}$$

It follows from (2.21) that

$$\limsup_{n\to\infty}K_1^{(n)}\le\kappa(f) \quad \text{a.s.} \tag{2.23}$$

From (2.5) and (2.6), it is easy to verify that

$$\lim_{n\to\infty}(K_2^{(n)}+K_3^{(n)})=0. \tag{2.24}$$

Combining (2.22), (2.23) and (2.24), we obtain (2.12).

Finally, we show (2.13). Let h_n be defined as in (2.1). The proof of (2.13) is very similar to that of (2.11), and hence, is omitted. The proof of Theorem 1.2 is now complete. □

Proof of Theorem 1.3 The proof of (1.9) is similar to that of Theorem 3 in [6]. So the details are omitted. The proof of Theorem 1.3 is completed. □

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关于 Wiener 过程泛函连续模的精确收敛速度

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摘 要: 设 $\{W(t), t\ge 0\}$ 是一标准 Wiener 过程, 记 S 是 Strassen 重对数律的紧集类. 本文中我们讨论了两个变量 $\sup_{0\le t\le 1-h}\inf_{f\in S}\sup_{0\le x\le 1}|(W(t+hx)-W(t))(2h\log h^{-1})^{-1/2}-f(x)|$ 及 $\inf_{0\le t\le 1-h}\sup_{0\le x\le 1}|(W(t+hx)-W(t))(2h\log h^{-1})^{-1/2}-f(x)|$(对任何 $f\in S$) 趋于零的精确的收敛速度. 作为一个推广, 我们建立了 Wiener 过程的不可微模与泛函的连续模之间的一种关系.