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C-Totally Real Submanifolds with Parallel Mean Curvature Vector of Sasakian Space Form *

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Abstract: We have discussed the C-totally real submanifolds with parallel mean curvature vector of Sasakian space form, obtained a formula of J.Simons type, and improved one result of S.Yamaguchi.

Key words: Sasakian space form; parallel mean curvature vector; C-totally real submanifold.

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1. Introduction

Let $\overline{M}^{2n+1}(k)$ be a (2n+1)-dimensional Sasakian manifold with constant ϕ-sectional curvature k, M^n be a compact and without boundary C-totally real submanfold immersed in $\overline{M}^{2n+1}(k)$. S and H respectively denoted the square of the norm of the second fundamental form and the mean curvature. S.Yamaguchi have studied C-totally real minimal submanifolds of $\overline{M}^{2n+1}(k)$, obtained a formula of J.Simons type and the following theorem(see [1])

Theorem A *Let $\overline{M}^{2n+1}(k)$ be a (2n+1)-dimensional Sasakian manifold with constant ϕ-holomorphic sectional curvature k, M^n a compact n-dimensional C-totally real minimal submanifold of $\overline{M}^{2n+1}(k)$. If $S \le \frac{1}{4}n(n+1)(k+3)/2n+1$, then M is totally geodesic.*

In this paper we discuss compact C-totally real submanifolds with parallel mean curvature vector field in a Sasakian space form. We shall prove the following theorems:

Theorem 1 *Let M^n be the compact n-dimensional C-totally real submanifold with parallel mean curvature vector field in a Sasakian space form $\overline{M}^{2n+1}(k)$, then*

$$\int_{M^n}\left[-\frac{1}{4}(n+1)(k+3)S+\frac{k+1}{2}n^2H^2+\frac{3}{2}\Phi^2+\frac{n(n-2)H}{\sqrt{n(n-1)}}\Phi^{\frac{3}{2}}-nH^2\Phi\right]*1\ge 0,$$

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where $\Phi = S - nH^2$.

Theorem 2 *Let M^n be a compact n-dimensional C-totally real minimal submanifold of a Sasakian space form $\overline{M}^{2n+1}(k)$. If $S < \frac{1}{6}(n+1)(k+3)$, then $M^n$is totaly geodesic.*

2. Preliminaries and lemmas

Let $\overline{M}^{2n+1}$ be a (2n+1)-dimensional ($n > 2$) Sasakian manifold with structure tensors (ϕ, ξ, η, g). Then the structure tensors satisfy the following equations:

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\phi X) = 0, \eta(X) = g(X, \xi), \eta(\xi) = 1,$$

$$g(\phi X, Y) + g(X, \phi Y) = 0, \nabla_X \xi = -\phi X, (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for any vector fields X, Y on $\overline{M}^{2n+1}$. A Sasakian manifold $\overline{M}^{2n+1}$ is a Sasakian space form if $\overline{M}^{2n+1}$ has constant ϕ-holomorphic sectional curvature k, and will be denoted by $\overline{M}^{2n+1}(k)$ and the curvature tensor is given by

$$\overline{R}(X,Y)Z = \frac{1}{4}(k+3)[g(Y,Z)X - g(X,Z)Y] + \frac{1}{4}(k-1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2g(X, \phi(Y))\phi(Z)]. \quad (1)$$

The Pfaffian equation $\eta = 0$ determines in $\overline{M}^{2n+1}$ a 2n-dimensional distribution. It is called the contact distribution. We shall call the integral submanifold M of the contact distribution of a Sasakian manifold a C-totally real submanifold. Then we have $\dim M \leq n$ and the following (see [2])

Theorem B *Let $M^m (m \leq n)$ be a C-totally real submanifold of a Sasakian manifold $\overline{M}^{2n+1}$. Then*

(i) the second fundamental form of ξ direction is identically zero;

(ii) if $X \in TM$, then $\phi X \in T^\perp(M)$;

(iii) for all $X, Y, Z \in TM, g(\phi X, B(Y, Z)) = g(\phi Y, B(X, Z))$.

Let M^n be a C-totally real submanifold of a Sasakin space form $\overline{M}^{2n+1}(k)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{n+1} = \phi e_1, \cdots, e_{2n} = \phi e_n, e_{2n+1} = \xi$ in $\overline{M}^{2n+1}(k)$ such that, restricted to M^n, the vectors $e_1, \cdots, e_n$ are tangent to M^n. We make the following convention on range of indices $1 \leq i, j, k, \cdots \leq n; n+1 \leq \alpha, \beta, \gamma, \cdots \leq 2n+1$.

Let $\omega^1, \cdots, \omega^{2n}, \omega^{2n+1} = \eta$ be the field of dual frames, we restricted these forms to M_n, then $\omega^\alpha = 0, \omega^\alpha_j = \sum h^\alpha_{ij}\omega^i, h^\alpha_{ij} = h^\alpha_{ji}$.

The Laplacian $\triangle h^\alpha_{ij}$ of the second fundamental form h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. A straightforward computation gives the following(see[3])

$$\sum h^\alpha_{ij} \triangle h^\alpha_{ij} = \sum (h^\alpha_{ij} h^\alpha_{kkij} - \overline{R}_{\alpha ij\beta} h^\alpha_{ij} h^\beta_{kk} + 4\overline{R}_{\alpha\beta ki} h^\alpha_{ij} h^\beta_{jk} - \overline{R}_{\alpha k\beta k} h^\alpha_{ij} h^\beta_{ij} + 2\overline{R}_{mkik} h^\alpha_{mj} h^\alpha_{ij} + 2\overline{R}_{mijk} h^\alpha_{mk} h^\alpha_{ij}) - \sum (h^\alpha_{ik} h^\beta_{jk} - h^\alpha_{jk} h^\beta_{ik})(h^\alpha_{il} h^\beta_{jl} - h^\alpha_{jl} h^\beta_{il}) - h^\alpha_{ij} h^\alpha_{kl} h^\beta_{ij} h^\beta_{kl} + h^\alpha_{ij} h^\alpha_{ki} h^\beta_{kj} h^\beta_{ll}.$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e, $N(A) = tr(A^t A) = \sum_{i,j} a^2_{ij}$.

Lemma 1[4] *Let $A_{n+1}, A_{n+2}, \cdots, A_{n+p}$ be $(n\times n)$-symmetric matrices and let $S_{\alpha\beta} = tr(A_\alpha^t A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, then $\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \le (1+\frac{1}{2}\mathrm{sgn}(p-1))(\sum_\alpha S_\alpha)^2$, where sgn(.) is the symbolic function.*

Lemma 2[5] *Let $A, B: R^n \to R^n$ be symmetric linear maps such that $[A,B]=0$ and $\mathrm{tr}A = \mathrm{tr}B = 0$. Then $-\frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{\frac{1}{2}} \le \mathrm{tr}A^2B \le \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{\frac{1}{2}}$.*

Lemma 3 *Let $\overline{M}^{2n+1}$ be a Sasakian space form, and M^n an n-dimensional C-totally real submanifold. The second fundamental form of M^n is $B = \sum h_{ij}^\alpha \omega^i \otimes \omega^j \otimes e_\alpha$, Then $h_{kj}^{n+i} = h_{ij}^{n+k}$.*

Proof From (iii)of the theorem B we have

$$h_{ij}^{n+k} = g(\phi e_k, B(e_i,e_j)) = g(\phi e_i, B(e_k,e_j)) = g(e_{n+i}, h_{kj}^\alpha e_\alpha) = h_{kj}^{n+i}.$$

3. Proof of Theorems

In the case that $H\neq 0$ we can choose a local field in such a way that $e_{n+1} = \phi e_1 = h/H$, then $trH_{n+1} = nH, trH_\alpha = 0(\alpha\neq n+1)$, where h is the mean curvature vector and $H_\alpha = (h_{ij}^\alpha)$. By use of (1),Theorem B and Lemma 3, we have

$$\begin{aligned}\sum h_{ij}^\alpha \triangle h_{ij}^\alpha = &\sum h_{ij}^\alpha h_{kkij}^\alpha + \frac{nk+3n+k-1}{4}S - \frac{k+1}{2}n^2H^2 - \\ &\sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta) - \sum h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta + \\ &\sum h_{ij}^\alpha h_{ki}^\alpha h_{kj}^\beta h_{ll}^\beta. \end{aligned}\tag{2}$$

Take $b_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}, b_{ij}^\alpha = h_{ij}^\alpha(\alpha = n+2,\cdots,2n+1), B_\alpha = (b_{ij}^\alpha)$, then we have $trB_\alpha = 0, \Phi := \sum_{\alpha,i,j}(b_{ij}^\alpha)^2 = S - nH^2$. Therefore from(2) we get

$$\begin{aligned}\sum h_{ij}^\alpha \triangle h_{ij}^\alpha = &\sum h_{ij}^\alpha h_{kkij}^\alpha + \frac{nk+3n+k-1}{4}S - \frac{k+1}{2}n^2H^2 - \\ &\sum (b_{ik}^\alpha b_{jk}^\beta - b_{jk}^\alpha b_{ik}^\beta)(b_{il}^\alpha b_{jl}^\beta - b_{jl}^\alpha b_{il}^\beta) - \sum b_{ij}^\alpha b_{kl}^\alpha b_{ij}^\beta b_{kl}^\beta + \\ &nH\sum b_{ij}^\alpha b_{ki}^\alpha b_{kj}^{n+1} + nH^2\sum (b_{ij}^\alpha)^2. \end{aligned}\tag{3}$$

From Lemma 1 we have $-\sum (b_{ik}^\alpha b_{jk}^\beta - b_{jk}^\alpha b_{ik}^\beta)(b_{il}^\alpha b_{jl}^\beta - b_{jl}^\alpha b_{il}^\beta) - \sum b_{ij}^\alpha b_{kl}^\alpha b_{ij}^\beta b_{kl}^\beta \ge -\frac{3}{2}(S-nH^2)^2$. Since $\nabla^\perp h = 0$, by use of Lemma 2, we get

$$nH\sum b_{ij}^\alpha b_{ki}^\alpha b_{kj}^{n+1} + nH^2\sum (b_{ij}^\alpha)^2 \ge nH\left[-\frac{n-2}{\sqrt{n(n-1)}}(S-nH^2)^{3/2}\right] + nH^2(S-nH^2).$$

On the other hand, we have $\sum h_{ij}^\alpha h_{kkij}^\alpha = 0$. Combining (3) and the above information, we conclude that

$$\sum h_{ij}^\alpha \triangle h_{ij}^\alpha \ge \frac{nk+3n+k-1}{4}S - \frac{k+1}{2}n^2H^2 - \frac{3}{2}\Phi^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}\Phi^{3/2} + nH^2\Phi.$$

Since $\langle \nabla^2 A, A\rangle = \sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha}$, and M^n are a compact submanifold, then(see[6])

$$\int_{M^n} \langle \nabla^2 A, A\rangle * 1 = -\int_{M^n} \langle \nabla A, \nabla A\rangle * 1.$$

Since (see [2]) $\langle \nabla A, \nabla A\rangle - |A|^2 \geq 0$, we have

$$\int_{M^n} \left(-\sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} - |A|^2\right) * 1 = \int_{M^n} \left(\langle \nabla A, \nabla A\rangle - |A|^2\right) * 1 \geq 0.$$

Hence

$$\int_{M^n} \left[-\frac{1}{4}(n+1)(k+3)S + \frac{k+1}{2}n^2H^2 + \frac{3}{2}\Phi^2 + \frac{n(n-2)H}{\sqrt{n(n-1)}}\Phi^{3/2} - nH^2\Phi\right] * 1 \geq 0. \quad (4)$$

We complete the proof of Theorem 1.

When M^n is an n-dimensional C-totally real minimal submanifoid, by use of the same method as before, we get

$$\int_{M^n} \left[-\frac{1}{4}(n-1)(k+3)S + \frac{3}{2}S^2\right] * 1 \geq 0. \quad (5)$$

From (5) we can easily get Theorem 2.

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Sasakian 空间形式中具有平行平均曲率向量的 C- 全实子流形

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摘 要: 本文讨论了 Sasakian 空间形式中具有平行平均曲率向量的 C- 全实子流形，得到了一个 Simons 型公式并且改进了 S.Yamaguchi 等的一个结果.