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Riemann-Hilbert Boundary Value Problem for the Cauchy-Riemann Equation *

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Abstract: The Riemann-Hilbert boundary value problem for the inhomogeneous Cauchy-Riemann equation in the polydomain is considered. The sufficient and necessary solvable conditions and integral expressions of solution for the above problem are given.

Key words: Riemann-Hilbert boundary value problem ; Cauchy-Riemann equation; holomorphic function.

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1. Introduction

There were some publications (see [1]-[4]), dedicated to extension the theories of the Riemann-Hilbert boundary value problems for a holomorphic function and a elliptic system and hyperbolic system and parabolic system of equation of a single complex variable and their generalizations to the case of a several complex variables .In the paper [5],we have discussed the Riemann-Hilbert boundary value problem for the elliptic system equation. Here we shall treat the Riemann-Hilbert boundary value problem for the in homogeneous Cauchy-Riemann equation .The discussion of this problem is little now.

2. Lemma and Problem

Let D_k be a bounded simply connected domain with the smooth boundary ∂D_k in the plane of a complex variable $z_k = x_k + iy_k$. The polydomain D is defined to be the set of all $z = (z_1 \cdots z_n) \in C^n$ such that $z_k \in D_k, k = 1, \cdots, n$. let $r(t_1 \cdots t_n)$ be Hölder continuous function given on distinguished boundary $\Gamma^n = \partial D_1 \times \cdots \partial D_n$ of D such that $r(t_1 \cdots t_n)$ is a real valued. First we consider the Riemann-Hilbert boundary value problem A for the holomorphic function: find in D a holomorphic function $w(z_1 \cdots z_n)$, continuous up to the Γ^n, satisfying the

$$\mathrm{Re}w(t_1 \cdots t_n) = r(t_1 \cdots t_n). \qquad (1)$$

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In the domain $\Gamma^n = \{(t_1 \cdots t_n) \in C^n, |t_i| = 1, i = 1 \cdots n\}$.

Lemma *The problem A for the holomorphic function is solvable if and only if $r(t_1 \cdots t_n)$ satisfies the conditions.*

$$\int_{|t_1|=1} \cdots \int_{|t_n|=1} r(t_1 \cdots t_n) t_1^{-\alpha_1 - 1} \cdots t_n^{-\alpha_n - 1} \mathrm{d}t_1 \cdots \mathrm{d}t_n = 0 \tag{2}$$

for $(\alpha_1 \cdots \alpha_n) \in \{(-\infty, +\infty) \cdots (-\infty, +\infty)\} - \{[0, +\infty) \cdots [0, +\infty)\} - \{(-\infty, 0] \cdots (-\infty, 0]\}$.

In this case the solution is given by

$$w(z_1 \cdots z_n) = \frac{1}{(2\pi i)^n} \int_{|t_1|=1} \cdots \int_{|t_n|=1} r(t_1 \cdots t_n) \left[\frac{2}{\prod_{i=1}^{n} (t_i - z_i)} - \frac{1}{\prod_{i=1}^{n} t_i} \right] \mathrm{d}t_1 \cdots \mathrm{d}t_n + ic. \tag{3}$$

Where C is an arbitrary real constant.

The proof of this lemma can see reference [6] please.

Next we consider the Riemann-Hilbert boundary value problem B for the inhomogeneous Cauchy-Riemann equation in the several bicylinder domain: find a solution $w(z_1 \cdots z_n)$ of the equations

$$w_{\overline{z_j}}(z_1 \cdots z_n) = f_j(z_1 \cdots z_n), \quad j = 1 \cdots n \tag{4}$$

in the domain $U^n = \{(z_1 \cdots z_n) \in C^n, |z_i| < 1, i = 1 \cdots n\}$ with given $f_j \in C_\alpha^1(\bar{U}^n)$, satisfying the boundary condition (1) on the torus

$$\Gamma^n = \{(t_1 \cdots t_n) \in C^n, |t_i| = 1, i = 1 \cdots n\}.$$

3. Result and proof

Theorem *The problem B for the equation (4) is solvable if and only if its right sides satisfy the conditions*

$$f_{j\bar{z}_i} = f_{i\bar{z}_j}, i \neq j, \quad i, j = 1 \cdots n, \tag{5}$$

$$\begin{aligned} &\frac{1}{2^{n-1}} \int_{|t_1|=1} \cdots \int_{|t_n|=1} r(t_1 \cdots t_n) t_1^{k_1} \cdots t_i^{-k_i} \cdots t_n^{k_n} \mathrm{d}t_1 \cdots \mathrm{d}\bar{t}_i \cdots \mathrm{d}t_n \\ &= \int_{|t_1|<1} \cdots \int_{|t_n|<1} \sum_{i=1}^{n} \left(\sum_{\substack{k=1 \\ k>i}}^{n} \frac{\partial \bar{f}_i}{\partial \bar{t}_k} + \sum_{\substack{k=1 \\ k<i}}^{n} \frac{\partial f_k}{\partial t_i} \right) t_1^{k_1} \cdots t_i^{-k_i} \cdots t_n^{k_n} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_n, \\ &\quad k_i \geq 0, i = 1 \cdots n. \end{aligned} \tag{6}$$

In this case the solution of the problem B is given by

$$w(z_1\cdots z_n)=-\frac{1}{\pi}\sum_{i=1}^{n}\int_{|t_i|<1}\left[\frac{f_i(z_1\cdots t_i\cdots z_n)}{t_i-z_i}+\frac{z_i f_i(z_1\cdots t_i\cdots z_n)}{1-z_i\bar{t}_i}\right]d\sigma_i-$$

$$\frac{1}{\pi^n}\int\limits_{|t_1|<1}\cdots\int\limits_{|t_n|<1}\left[\frac{1}{\prod_{i=1}^{n}(t_i-z_i)}\sum_{i=1}^{n}\sum_{\substack{k=1\\k>i}}^{n}\frac{\partial f_i(t_1\cdots t_n)}{\partial\bar{t}_k}-\right.$$

$$\left.\frac{z_1\cdots z_n}{\prod_{i=1}^{n}(1-z_i\bar{t}_i)}\sum_{i=1}^{n}\sum_{\substack{k=1\\k>i}}^{n}\frac{\partial\overline{f_i(t\cdots t)}}{\partial t_k}\right]d\sigma_1\cdots d\sigma_n-$$

$$\frac{1}{\pi^n}\int\limits_{|t_1|<1}\cdots\int\limits_{|t_n|<1}\sum_{i=1}^{n}\left[\frac{z_i}{(1-z_i\bar{t}_i)\prod_{\substack{k=1\\k\neq i}}^{n}(t_k-z_k)}\sum_{\substack{k=1\\k\neq i}}^{n}\frac{\partial\overline{f_i(t_1\cdots t_n)}}{\partial\bar{t}_k}\right]d\sigma_1\cdots d\sigma_n+$$

$$\frac{1}{(2\pi)^n}\int\limits_{|t_1|=1}\cdots\int\limits_{|t_n|=1}r(t_1\cdots t_n)\left(\frac{2}{\prod_{i=1}^{n}(1-z_i\bar{t}_i)}-1\right)d\theta_1\cdots d\theta_n+iC. \tag{7}$$

Where C is an arbitrary real constant.

Proof Integrating (4) and using (5) we obtain

$$w(z_1\cdots z_n)=\varphi(z_1\cdots z_n)-\frac{1}{\pi}\sum_{i=1}^{n}\int\limits_{|t_i|<1}\left[\frac{f_i(z_1\cdots t_i\cdots z_n)}{t_i-z_i}\right]d\sigma_i+$$

$$\frac{1}{\pi^n}\int\limits_{|t_1|<1}\cdots\int\limits_{|t_n|<1}\frac{1}{\prod_{i=1}^{n}(t_i-z_i)}\sum_{i=1}^{n}\sum_{\substack{k=1\\i<k}}^{n}\frac{\partial f_i(t_1\cdots t_n)}{\partial\bar{t}_k}d\sigma_1\cdots d\sigma_n, \tag{8}$$

where $\varphi(z_1\cdots z_n)$ is an arbitrary holomorphic function in U^n. Hence (1) leads to the condition for the holomorphic function $\varphi(z_1\cdots z_n)$

$$\mathrm{Re}\varphi(t_1\cdots t_n)=r_0(t_1\cdots t_n),|t_i|=1,i=1\cdots n. \tag{9}$$

With

$$r_0(z_1\cdots z_n)=r(z_1\cdots z_n)+\mathrm{Re}\frac{1}{\pi}\sum_{i=1}^{n}\int_{|t_i|<1}\left[\frac{f_i(z_1\cdots t_i\cdots z_n)}{t_i-z_i}\right]d\overline{\sigma_i}-$$

$$\mathrm{Re}\frac{1}{\pi^n}\int_{|t_1|<1}\cdots\int_{|t_n|<1}\frac{1}{\prod_{i=1}^{n}(t_i-z_i)}\sum_{i=1}^{n}\sum_{\substack{k=1\\i<k}}^{n}\frac{\partial f_i(t_1\cdots t_n)}{\partial\bar{t}_k}d\overline{\sigma_1}\cdots d\overline{\sigma_n}. \tag{10}$$

The condition (2) and formula (3) yields (6) and (7) .Now we show that if (5) and (6) hold , then (8) is the solution of the problem (4) and (1). By direct calculation and taking into account the compatibility condition (5). it is an easy to derive from (7) that $w_{\overline{z_j}} = f_j(z_1 \cdots z_n), j = 1 \cdots n$ for $(z_1 \cdots z_n) \in \overline{U}^n$. Moreover from (7) we have

$$\begin{aligned}
&\operatorname{Re} w(z_1 \cdots z_n) \\
&= \operatorname{Re}\Big\{ -\frac{1}{\pi}\sum_{i=1}^{n} \int\limits_{|t_i|<1} \left[\frac{f_i(z_1 \cdots t_i \cdots z_n)}{t_i - z_i} + \frac{z_i \overline{f_i(z_1 \cdots t_i \cdots z_n)}}{1 - z_i\overline{t_i}}\right] \mathrm{d}\sigma_i - \\
&\frac{1}{\pi^n} \int\limits_{|t_1|<1} \cdots \int\limits_{|t_n|<1} \left[\frac{1}{\prod\limits_{i=1}^{n}(t_i - z_i)} \sum_{i=1}^{n}\sum_{\substack{k=1\\k>i}}^{n} \frac{\partial f_i(t_1 \cdots t_n)}{\partial \overline{t_k}} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_n\right] - \\
&\frac{1}{\pi^n} \int\limits_{|t_1|<1} \cdots \int\limits_{|t_n|<1} \frac{z_1 \cdots z_n}{\prod\limits_{i=1}^{n}(1 - z_i\overline{t_i})} \sum_{i=1}^{n}\sum_{\substack{k=1\\k>i}}^{n} \overline{\frac{\partial f_i(t_1 \cdots t_n)}{\partial \overline{t_k}}} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_n \Big\} + \\
&\operatorname{Re}\Big\{ \frac{1}{(2\pi)^n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} r(t_1 \cdots t_n) \left[\frac{2}{\prod\limits_{i=1}^{n}(1 - z_i\overline{t_i})} - 1\right] \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n - \\
&\sum_{i=1}^{n} \frac{z_i}{\pi^n} \int\limits_{|t_1|<1} \cdots \int\limits_{|t_n|<1} \frac{1}{(1 - z_i\overline{t_i}) \prod\limits_{\substack{k=1\\k\neq i}}^{n}(t_k - z_k)} \sum_{\substack{k=1\\k\neq i}}^{n} \overline{\frac{\partial f_i(t_1 \cdots t_n)}{\partial t_k}} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_n \Big\},
\end{aligned}$$

but

$$\begin{aligned}
&\operatorname{Re}\frac{1}{(2\pi)^n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} r(t_1 \cdots t_n) \left[\frac{2}{\prod\limits_{i=1}^{n}(1 - z_i\overline{t_i})} - 1\right] \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\
&= \frac{1}{(2\pi)^n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} \frac{\prod\limits_{i=1}^{n}\left(1 - |z_i|^2\right) r(t_1 \cdots t_n)}{\prod\limits_{i=1}^{n}|1 - z_i\overline{t_i}|^2} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n + \\
&\frac{2}{(2\pi)^n}\operatorname{Re}\sum_{i=1}^{n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} \frac{r(t_1 \cdots t_n)}{1 - z_i\overline{t_i}} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n - \\
&\frac{2}{(2\pi)^n}\operatorname{Re} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} \frac{r(t_1 \cdots t_n)}{\prod\limits_{i=1}^{n}(1 - z_i\overline{t_i})} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n - \\
&\frac{2}{(2\pi)^n} \int\limits_{|t_1|=1} \cdots \int\limits_{|t_n|=1} r(t_1 \cdots t_n)\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n
\end{aligned}$$

$$= \frac{1}{2\pi}\int\limits_{|t_1|=1}\cdots\Big[\frac{1}{2\pi}\int\limits_{|t_{n-1}|=1}\left(\frac{1}{2\pi}\int\limits_{|t_n|=1} r(t_1\cdots t_n)\frac{1-|z_n|^2}{1+|z_n|^2-2\mathrm{Re}z_n\overline{t_n}}\mathrm{d}\theta_n\right)\cdot$$

$$\frac{1-|z_{n-1}|^2}{1+|z_{n-1}|^2-2\mathrm{Re}z_{n-1}\overline{t_{n-1}}}\mathrm{d}\theta_{n-1}\cdots\Big]\frac{1-|z_1|^2}{1+|z_1|^2-2\mathrm{Re}z_1\overline{t_1}}\mathrm{d}\theta_1- \tag{11}$$

$$\mathrm{Re}\sum_{\substack{k_i>0\\ i=1\cdots n}}\frac{z_1^{k_1}\cdots z_i^{-k_i}\cdots z_n^{k_n}}{(2\pi)^n}\int\limits_{|t_1|=1}\cdots\int\limits_{|t_n|=1} r(t_1\cdots t_n)t_1^{-k_1}\cdots t_i^{k_i}\cdots t_n^{-k_n}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n.$$

Hence when $|t_i| = 1, i = 1\cdots n$, the first of three terms in (11) vanishes immediately, because of under the real sign is a pure imaginary quantity and the last two terms can be written as

$$-\mathrm{Re}\Big\{\frac{z_1\cdots\overline{z_i}\cdots z_n}{\pi^n}\int\limits_{|t_1|<1}\cdots\int\limits_{|t_n|<1}\frac{1}{\prod\limits_{i=1}^{n}(1-z_i\overline{t_i})}\sum_{i=1}^{n}\Big[\sum_{\substack{k=1\\ k>i}}^{n}\overline{\frac{\partial f_i(t_1\cdots t_n)}{\partial t_k}}+$$

$$\sum_{\substack{k=1\\ k<i}}^{n}\frac{\partial f_i(t_1\cdots t_n)}{\partial t_k}\Big]\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_n\Big\}$$

$$=-\mathrm{Re}\Big\{\sum_{\substack{k_i>0\\ i=1\cdots n}}\frac{z_1^{k_1+1}\cdots z_i^{-k_i+1}\cdots z_n^{k_n+1}}{\pi^n}\int\limits_{|t_1|<1}\cdots\int\limits_{|t_n|<1}\sum_{i=1}^{n}\Big[\sum_{\substack{k=1\\ k>i}}^{n}\overline{\frac{\partial f_i(t_1\cdots t_n)}{\partial t_k}}+$$

$$\sum_{\substack{k=1\\ k<i}}^{n}\frac{\partial f_i(t_1\cdots t_n)}{\partial t_k}\Big]t_1^{-k_1}\cdots t_i^{k_i}\cdots t_n^{-k_n}\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_n\Big\}$$

Hence taking into account (6) and the boundary values of poisson integrals for the domain $|z_i| < 1, i = 1\cdots n$, we obtain

$$\mathrm{Re}w(z_1\cdots z_n) = \frac{1}{2\pi}\int\limits_{|t_1|=1}\Big\{\cdots\int\limits_{|t_{n-1}|=1}\Big[\frac{1}{2\pi}\int\limits_{|t_n|=1} r(t_1\cdots t_n)\frac{1-|z_n|^2}{1+|z_n|^2-2\mathrm{Re}z_n\overline{t_n}}\mathrm{d}\theta_n\cdot$$

$$\frac{1-|z_{n-1}|^2}{1+|z_{n-1}|^2-2Rez_{n-1}\overline{t_{n-1}}}\Big]\mathrm{d}\theta_{n-1}\cdots\Big\}\frac{1-|z_1|^2}{1+|z_1|^2-2\mathrm{Re}z_1\overline{t_1}}\mathrm{d}\theta_1$$

$$\to r(t_1\cdots t_n),(t_1\cdots t_n)\in\Gamma^n.$$

We have the conclusion .

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Cauchy-Riemann 方程的 Riemann-Hilbert 边值问题

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摘 要: 本文考虑多柱域上非齐次的 Cauchy-Riemann 方程的 Riemann-Hilbert 边值问题. 讨论了上述边值问题可解的充分必要条件，并给出了边值问题解的积分表达式.