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Necessary and Sufficient Conditions for Oscillation of Bounded Solutions of Nonlinear Second Order Difference Equations *

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Abstract: In this paper, we give necessary and sufficient conditions for oscillation of bounded solutions of nonlinear second order difference equation $\Delta(p_n\Delta y_n) + q_n f(y_{n-r_n}) = 0$. Obtained results improve theorems in the literature [3,6,7].

Key words: second order difference equation; oscillation, delay.

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1. Introduction

Consider the nonlinear second order difference equation

$$\Delta(p_n\Delta y_n) + q_n f(y_{n-r_n}) = 0, \tag{1}$$

where $\{p_n\}$is a sequence of positive numbers, $P_n = \sum\limits_{s=1}^{n-1}\frac{1}{p_s} \to \infty$, as $n \to \infty$, $\{q_n\}$ is a sequence of real numbers, $\{r_n\}$ is a sequence of positive integer,and $\lim\limits_{n\to\infty}(n - r_n) = \infty$, $\Delta y_n = y_{n+1} - y_n$, $f \in C(R, R)$, and $\frac{f(u)}{u} \geq 1$ as $u \neq 0$. A solution y_n of Eq(1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

In many papers, oscillation for solutions of nonlinear second order difference equation has been investigated[1-7] .In this paper we give necessary and sufficient conditions for oscillation of bounded solutions of nonlinear second order difference equations, obtained results obtained improve theorems in the literature[3,6,7].

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2. Main results

Theorem 1 *Assume that $q_n \geq 0$ and*

$$\sum_{k=0}^{\infty} q_n = \infty. \tag{2}$$

Then every solution of (1) is oscillatory.

Proof Suppose the contrary, assume y_n is an eventually positive solution of (1). Then there exists a positive integer n_0 , such that

$$y_{n-r_n} > 0 \quad \text{as} \quad n \geq n_0. \tag{3}$$

In view of Eq(1), we have

$$\Delta(p_n \Delta y_n) = -q_n f(y_{n-r_n}) \leq 0 \quad \text{as} \quad n \geq n_0,$$

then $\{p_n \Delta y_n\}$ is eventually nonincreasing sequence, we shall show that

$$p_n \Delta y_n \geq 0 \quad \text{as} \quad n \geq n_0. \tag{4}$$

In fact, if not, then there exists $n_1 \geq n_0$, such that $p_{n_1} \Delta y_{n_1} = -c < 0$ and $p_n \Delta y_n \leq -c \quad n \geq n_1$, hence,

$$y_n \leq y_{n_1} - c \sum_{k=n_1}^{n-1} \frac{1}{p_k} \to -\infty \quad \text{as} \quad n \to \infty,$$

which is contradictory to the fact that y_n is eventually positive solution. Therefore (3) satisfies, thus we have

$$y_{n-r_n} > 0, \quad \Delta y_n \geq 0, \quad \triangle(p_n \Delta y_n) \leq 0 \quad \text{as} \quad n \geq n_0.$$

Hence there exists $n_1 \geq n_0$ and $l > 0$ such that

$$f(y_{n-r_n}) \geq y_{n-r_n} \geq l \quad \text{as} \quad n \geq n_1. \tag{5}$$

Substituting (5) into (1), we obtain

$$\triangle(p_n \Delta y_n) + l q_n \leq 0 \quad \text{as} \quad n \geq n_1. \tag{6}$$

Summing (6), we have

$$p_{n+1} \Delta y_{n+1} - p_{n_1} \Delta y_{n_1} + l \sum_{k=n_1}^{n} q_k \leq 0.$$

From the above and (3), we have

$$\sum_{k=0}^{\infty} q_k \leq \frac{p_{n_1} \Delta y_{n_1}}{l}.$$

which contradicts (2). The proof is complete.

Remark 1 Theorem 1 extends relevant results of [3].

Theorem 2 *Assume that $q_n \geq 0$, $f(u)$ is nondecreasing. The every bounded solution of (1) is oscillatory if and only if*

$$\sum_{n=1}^{\infty} P_n q_{n-1} = \infty. \tag{7}$$

Proof Sufficiency. Assume y_n is bounded eventually positive solution of (1), similar to the proof of Theorem 1, we can obtain inequality (6). Hence

$$P_{n+1}\Delta\left(p_n \Delta y_n\right) + P_{n+1} l q_n \leq 0, n \geq n_1. \tag{8}$$

It is not difficult to see that

$$P_{n+1}\Delta\left(p_n \Delta y_n\right) = \Delta\left(P_n p_n \Delta y_n\right) - p_n \Delta y_n \Delta P_n. \tag{9}$$

Since $p_n \Delta P_n \equiv 1$, combining (8) and (9), we have

$$\sum_{k=n_1}^{n} \Delta\left(P_k p_k \Delta y_k\right) - \sum_{k=n_1}^{n} \Delta y_k + l \sum_{k=n_1}^{n} P_{k+1} q_k \leq 0, n \geq n_{1_1}.$$

From the above, we have

$$l \sum_{k=n_1}^{n} P_{k+1} q_k \leq y_{n+1} - y_{n_1} + P_{n_1} p_{n_1} \Delta y_{n_1}, n \geq n_1.$$

Since y_n is a bounded solution of (1), Then $\sum\limits_{k=n_1}^{\infty} P_{k+1} q_k < \infty$ which contradicts the assumption of Theorem.

Necessity. If $\sum_{n=1}^{\infty} P_n q_{n-1} < \infty$, we can show Eq(1) has a bounded eventually positive solution. There exists $N \geq n_0$, such that $\sum\limits_{s=n}^{\infty} P_s q_{s-1} \leq \frac{1}{2f(2)}$ for $n \geq N$.
Let

$$y_{n,0} \equiv 2,$$

$$y_{n,k+1} = \begin{cases} 1 + \sum\limits_{s=N}^{n-1} P_s q_{s-1} f\left(y_{s-1-\tau_{s-1},k}\right) + P_n \sum\limits_{n-1}^{\infty} q_s f\left(y_{s-\tau_s,k}\right), & n > N, \\ y_{N+1,k+1}, & N - \tau_N \leq n \leq N. \end{cases} \tag{10}$$

$$y_{n,1} = \begin{cases} 1 + f(2)\left(\sum\limits_{s=N}^{n-1} P_s q_{s-1} + P_n \sum\limits_{n-1}^{\infty} q_s\right) \leq y_{n,0} = 2, & n > N, \\ y_{N+1,1} \leq y_{N+1,0} = 2, & \tau_N \leq n \leq N. \end{cases}$$

Assume, $1 \le y_{n,k} \le y_{n,k-1} \le y_{n,0} = 2$, for $n \ge N - \tau_N$. Then,

$$1 \le y_{n,k+1} = \begin{cases} 1 + \sum\limits_{s=N}^{n-1} P_s q_{s-1} f\left(y_{s-1-\tau_{s-1}}\right) + P_n \sum\limits_{n-1}^{\infty} q_s f\left(y_{s-\tau_s}\right) \le y_{n,k}, & n > N, \\ y_{N+1,k+1} \le y_{N+1,k}, & N - \tau_N \le n \le N. \end{cases}$$

Therefore for any positive integer k, we can know $1 \le y_{n,k} \le y_{n,k-1} \le y_{n,0} = 2$, for $n \ge N - \tau_N$. Hence the limit of $y_{n,k}$ exists, i.e., $\lim\limits_{k\to\infty} y_{n,k} = y_n$ exists and $1 \le y_n \le 2$, for $n \ge N - \tau_N$, by the Lebesgue's Dominated Convergence Theorem in (10) we can obtain

$$y_n = \begin{cases} 1 + \sum\limits_{s=N}^{n-1} P_s q_{s-1} f\left(y_{s-1-\tau_{s-1}}\right) + P_n \sum\limits_{n-1}^{\infty} q_s f\left(y_{s-\tau_s}\right), & n > N, \\ y_{N+1}, & N - \tau_N \le n \le N. \end{cases}$$

Then for this y_n we can see $\Delta\left(p_n \Delta y_n\right) = -q_n f\left(y_{n-\tau_n}\right)$, for $n \ge N$, i.e., y_n is bounded eventually positive solution of (1). The proof is complete.

Theorem 3 *Assume that $q_n \ge 0$, f(u) is nondecreasing. Then every bounded solution of (1) is oscillatory if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \sum_{s=n}^{\infty} q_s = \infty. \tag{11}$$

Proof Sufficiency. Assume y_n is bounded eventually positive solution of (1). similar to the proof of Theorem 1, we can obtain inequality

$$\Delta\left(p_n \Delta y_n\right) + l q_n \le 0, \quad n \ge n_1.$$

Summing the above inequality, we have

$$p_{n+m+1} \Delta y_{n+m+1} - p_n \Delta y_n + l \sum_{s=n}^{n+m} q_s \le 0.$$

Since (4) and let $m \to \infty$,we have

$$l \frac{1}{p_n} \sum_{s=n}^{\infty} q_s \le \Delta y_n. \tag{12}$$

Summing (12) from N to k, we have $l \sum_{n=N}^{k} \frac{1}{p_n} \sum_{s=n}^{\infty} q_s \le y_{k+1}$, hence $\sum_{n=N}^{\infty} \frac{1}{p_n} \sum_{s=n}^{\infty} q_s < \infty$, which contradicts the assumption of theorem.

Necessity. If $\sum_{n=1}^{\infty} P_n q_{n-1} < \infty$,we can show Eq(1) has a bounded eventually positive solution. There exists $N \ge n_0$, such that,

$$\sum_{i=n}^{\infty} \frac{1}{p_i} \sum_{s=i}^{\infty} q_s < \frac{1}{2 f(2)}.$$

Let

$$y_{n,0} \equiv 2,$$

$$y_{n,k+1} = \begin{cases} 1 + \sum\limits_{i=N}^{n-1} \frac{1}{p_i} \sum\limits_{s=i}^{\infty} q_s f\left(y_{s-\tau_s,k}\right), & n > N, \\ y_{N+1,k+1}, & N - \tau_N \le n \le N. \end{cases} \quad k = 1, 2, \cdots$$

By mathematical induction similar to the proof in Theorem 1, we have

$$1 \le y_{n,k} \le y_{n,k-1} \cdots y_{n,1} \le y_{n,0} = 2, \quad n \ge N - \tau_N.$$

Hence $\lim_{k\to\infty} y_{n,k} = y_n$ exists and $1 \le y_n \le 2, n \ge N - \tau_N$, by the Lebesgue's Dominated Convergence Theorem, we obtain

$$y_n = \begin{cases} 1 + \sum\limits_{i=N}^{n-1} \frac{1}{p_i} \sum\limits_{s=i}^{\infty} q_s f\left(y_{s-\tau_s}\right), & n > N, \\ y_{N+1}, & N - \tau_N \le n \le N. \end{cases}$$

Then for this y_n, we have

$$\Delta\left(p_n \Delta y_n\right) = -q_n f\left(y_{n-\tau_n}\right), \quad n \ge N,$$

i.e, y_n is bounded eventually positive solution of (1). The proof is completed.
From Theorem 2 or Theorem 3, we can derive the following comparison result.
Consider comparison equation

$$\Delta\left(p'_n \Delta y_n\right) + q'_n f\left(y_{n-\tau_n}\right) = 0, \tag{13}$$

where $p'_n > 0, P'_n = \sum_{s=1}^{n-1} \frac{1}{p'_s} \to \infty$, as, $n \to \infty$, $q'_n \ge 0$.

Theorem 4 (Comparison Theorem) *Assume $p_n \le p'_n$, $q_n \ge q'_n$. Then that every bounded solution of (13) is oscillatory implies the same to (1).*

Remark 2 Theorem 2 and Theorem 3 extend and improve relevant results of [6,7] respectively. In particular, the conditions (7) and (11) that we have given are necessary and sufficient conditions respectively.

Example 1 Consider the nonlinear second order difference equation:

$$\Delta\left(\sqrt{n}\Delta y_n\right) + \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} f\left(y_{n-4}\right) = 0, \tag{14}$$

where $f \in c(R, R)$, $\frac{f(u)}{u} \ge \varepsilon > 0$, $(u \ne 0)$, and $f(u)$ is nondecreasing. Then

$$\sum_{s=n}^{\infty} q_s = \sum_{s=n}^{\infty} \frac{1}{(s+1)\sqrt{s} + s\sqrt{s+1}} = \frac{1}{\sqrt{n}},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \sum_{s=n}^{\infty} q_s = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

By Theorem 3, every bounded solution of (14) is oscillatory.

Example 2 Consider the difference equation:

$$\Delta\left((n+1)\,\Delta y_n\right)+\frac{n-1}{(n-2)\,n\,(n+1)}y_{n-1}=0,\quad n>2. \tag{15}$$

Since

$$q_n=\frac{n-1}{(n-2)\,n\,(n+1)}\leq\frac{1}{n\,(n+1)}\quad\text{and}\quad\sum_{s=n}^{\infty}\frac{1}{s\,(s+1)}=\frac{1}{n}$$

Hence (11) does not hold, by Theorem3 Eq(15) has bounded eventually positive solution. In fact, $y_n=\frac{n-1}{n}$ is such a solution.

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二阶非线性差分方程有界解振动的充分必要条件

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摘 要: 在本文中, 我们给出了非线性二阶差分方程 $\Delta(p_n\Delta y_n)+q_nf(y_{n-\tau_n})=0$ 有界解振动的充分必要条件和比较定理, 所得结果推广了文 [3,6,7] 的相应定理 .