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Comparison of Some Means *

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Abstract: The known results on comparison of extended mean values are used to compare power means, Stolarsky means and Heron mean. Some proofs of well known results are simplified with several new results obtained.

Key words: extended mean value; power mean; Stolarsky mean; Heron mean

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Let x and y be two positive real numbers.

The power mean is defined by $A_p(x,y)=\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}$ for $p\neq 0$, with

$$A_0(x,y)=\lim_{p\to 0}A_p(x,y)=\sqrt{xy}=G(x,y).$$

Clearly $A_1(x,y)=\frac{x+y}{2}=A(x,y)$.

The Stolarsky mean is defined by $S_p(x,y)=\left[\frac{x^p-y^p}{p(x-y)}\right]^{\frac{1}{p-1}}$ for $p\neq 0,1$, with

$$S_0(x,y)=\lim_{p\to 0}S_p(x,y)=\frac{x-y}{\ln x-\ln y}=L(x,y),$$

$$S_1(x,y)=\lim_{p\to 1}S_p(x,y)=e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}=I(x,y).$$

Both power means and Stolarsky means are special cases of the so-called extended mean values $E(r,s;x,y)$ defined as follows:

$$E(r,s;x,y)=\left[\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right]^{\frac{1}{s-r}},\qquad rs(r-s)(x-y)\neq 0;$$

$$E(r,0;x,y)=\left[\frac{1}{r}\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}},\qquad r(x-y)\neq 0;$$

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$$E(r,r;x,y) = e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}, \quad r(x-y)\neq 0;$$

$$E(0,0;x,y) = \sqrt{xy}, \quad x \neq y;$$

$$E(r,s;x,x) = x, \quad x = y.$$

In fact $E(p,2p;x,y) = E(2p,p;x,y) = A_p(x,y), E(1,p;x,y) = E(p,1;x,y) = S_p(x,y)$.

Make a comparison between power means and Stolarsky means is interesting. Stolarsky himself[1], Vamanamurthy and Vuorinen[2], have done this work one after another.

The purpose of this article is to use some known results on comparison of extended mean values to compare power means, Stolarsky means and Heron mean. 3. 3. Theorem of [2] and Theorem 2 of [1] are proved in a different and simpler way. Some new results are obtained, especially we have got the inequalities $G(x,y) \leq L(x,y) \leq H_e(x,y) \leq I(x,y) \leq A(x,y)$.

Lemma 1[3] *$E(r,s;x,y)$ increases with increase in either r or s.*

Lemma 2[4] *Let r,s,u,v be real numbers with $r\neq s, u\neq v$. Then inequality*

$$E(r,s;a,b) \leq E(u,v;a,b) \tag{1}$$

is satisfied for all $a,b>0$ if and only if

$$r+s \leq u+v, \tag{2}$$

$$e(r,s) \leq e(u,v), \tag{3}$$

where

$$e(x,y) = \begin{cases} \frac{x-y}{\ln(x/y)}, & \text{for } xy>0, x\neq y, \\ 0, & \text{for } xy=0 \end{cases} \tag{4}$$

if either $0 \leq \min\{r,s,u,v\}$ or $\max\{r,s,u,v\} \leq 0$; and

$$e(x,y) = \frac{|x|-|y|}{x-y} \text{ for } x,y\in \mathbf{R}, x\neq y \tag{5}$$

if $\min\{r,s,u,v\} < 0 < \max\{r,s,u,v\}$.

Theorem 1 *Let x,y be positive and distinct. Then $S_p(x,y) \leq A_p(x,y)$ for $p\in[\frac{1}{2},+\infty)$ with equality holds if and only if $p=\frac{1}{2}$, $S_p(x,y) > A_p(x,y)$ for each $p\in(-\infty,\frac{1}{2})$.*

Proof It is immediate from Lemma 1 and observe that $S_{\frac{1}{2}}(x,y) = A_{\frac{1}{2}}(x,y)$.

Theorem 2 *Let x,y be positive and distinct. Then $S_{p+1}(x,y) \leq A_p(x,y)$ for $p\in[1,+\infty)$ with equality holds if and only if $p=1$, $S_{p+1}(x,y) > A_p(x,y)$ for each $p\in(-\infty,1)$.*

Proof Notice that

$$S_{p+1}(x,y) = E(p+1,1;x,y) \text{ and } A_p(x,y) = E(p,2p;x,y),$$

we need only to check the condition (3).

If $p > 0$, then $\min\{p+1, 1, p, 2p\} > 0$. So by (4),

$$e(p+1,1) = \frac{p}{\ln(p+1)} \text{ and } e(p,2p) = \frac{p}{\ln 2}.$$

Clearly, $e(p+1,1) \le e(p,2p)$ if and only if $p \ge 1$, and $e(p+1,1) > e(p,2p)$ if and only if $p < 1$.

If $p < 0$, then $\min\{p+1,1,p,2p\} < 0 < \max\{p+1,1,p,2p\}$. So by (5),

$$e(p+1,p) = \begin{cases} 1, & \text{for } -1 \le p < 0, \\ -\frac{p+2}{2}, & \text{for } p < -1, \end{cases}$$

and

$$e(p,2p) = -1.$$

It follows $e(p+1,1) > e(p,2p)$.

Thus the theorem is proved by Lemma 2 and observe that $S_1(x,y) > A_0(x,y)$ and $S_2(x,y) = A(x,y)$.

Remark 1 Theorem 1 and Theorem 2 generalize the 3. 3. Theorem of [2] and the proofs are different and simpler.

Theorem 3 *Let x, y be positive and distinct Then $S_{p-1}(x,y) \le A_p(x,y)$ for $p \in [0,+\infty)$ with equality holds if and only if $p = 0$, $S_{p-1}(x,y) > A_p(x,y)$ for each $p \in (-\infty, 0)$.*

Proof Notice that

$$S_{p-1}(x,y) = E(p-1,1;x,y) \text{ and } A_p(x,y) = E(p,2p;x,y)$$

we need only to check the condition (3).

If $p \ge 1$ and $p \ne 2$, then $\min\{p-1,1,p,2p\} \ge 0$. So by (4),

$$e(p-1,1) = \begin{cases} \frac{p-2}{\ln(p-1)}, & \text{for } p > 1 \text{ and } p \ne 2, \\ 0, & \text{for } p = 1 \end{cases}$$

$$e(p,2p) = \frac{p}{\ln 2}.$$

Let $f(p) = (p-2)\ln 2 - p\ln(p-1)$ for $p > 1$. Then

$$f'(p) = \ln 2 - \ln(p-1) - \frac{p}{p-1},$$
$$f''(p) = \frac{2-p}{(p-1)^2}.$$

We have $f(2) = 0$, $f'(2) = \ln 2 - 2 < 0$ and $f''(2) = 0$. Since $f''(p) > 0$ for $1 < p < 2$ and $f''(p) < 0$ for $p > 2$, we can deduce that $f'(p) < 0$ for $p > 1$. It follows $f(p) > 0$ for $1 < p < 2$ and $f(p) < 0$ for $p > 2$. i.e., $(p-2)\ln 2 > p\ln(p-1)$ for $1 < p < 2$ and

$(p-2)\ln 2 < p\ln(p-1)$ for $p > 2$. Consequently, $\frac{p-2}{\ln(p-1)} < \frac{p}{\ln 2}$ for $p > 1$ and $p \neq 2$. Hence $e(p-1,1) < e(p,2p)$ for $p \geq 1$ and $p \neq 2$.

If $0 < p < 1$, then $\min\{p-1,1,p,2p\} < 0 < \max\{p-1,1,p,2p\}$. So by (5),

$$e(p-1,1) = \frac{p}{2-p} \text{ and } e(p,2p) = 1.$$

Hence $e(p-1,1) < e(p,2p)$ for $0 < p < 1$.

If $p < 0$ then $\min\{p-1,1,p,2p\} < 0 < \max\{p-1,1,p,2p\}$. So by (5),

$$e(p-1,1) = \frac{p}{2-p} \text{ and } e(p,2p) = -1.$$

Hence $e(p-1,1) > e(p,2p)$ for $p < 0$.

Thus the theorem is proved by Lemma 2 and observe that $S_1(x,y) < A(x,y)$ and $S_{-1}(x,y) = A_0(x,y)$.

Theorem 4 *Let x,y be positive and distinct. Then $S_p(x,y) \leq A_{\frac{p+1}{3}}(x,y)$ for $p \in [-1,\frac{1}{2}] \cup [2,+\infty)$ with equality holds if and only if $p = -1, \frac{1}{2}$ or 2, $S_p(x,y) > A_{\frac{p+1}{3}}(x,y)$ for each $p \in (-\infty,-1) \cup (\frac{1}{2},2)$.*

Proof Notice that

$$S_p(x,y) = E(p,1;x,y) \text{ and } A_{\frac{p+1}{3}}(x,y) = E\left(\frac{p+1}{3}, \frac{2(p+1)}{3}; x,y\right),$$

we need only to check the condition (3).

If $p < -1$, then $\min\{p,1,\frac{p+1}{3},\frac{2(p+1)}{3}\} < 0 < \max\{p,1,\frac{p+1}{3},\frac{2(p+1)}{3}\}$. So by (5),

$$e(p,1) = \frac{p+1}{1-p} \text{ and } e\left(\frac{p+1}{3}, \frac{2(p+1)}{3}\right) = -1.$$

Hence $e(p,1) > e\left(\frac{p+1}{3}, \frac{2(p+1)}{3}\right)$ for $p < -1$.

If $-1 < p < 0$, then $\min\{p,1,\frac{p+1}{3},\frac{2(p+1)}{3}\} < 0 < \max\{p,1,\frac{p+1}{3},\frac{2(p+1)}{3}\}$. So by (5),

$$e(p,1) = \frac{p+1}{1-p} \text{ and } e\left(\frac{p+1}{3}, \frac{2(p+1)}{3}\right) = 1.$$

Hence $e(p,1) < e\left(\frac{p+1}{3}, \frac{2(p+1)}{3}\right)$ for $-1 < p < 0$.

If $p > 0$ and $p \neq 1$, then $\min\{p,1,\frac{p+1}{3},\frac{2(p+1)}{3}\} \geq 0$. So by (4),

$$e(p,1) = \frac{p-1}{\ln p} \text{ and } e\left(\frac{p+1}{3}, \frac{2(p+1)}{3}\right) = \frac{p+1}{3\ln 2}.$$

Let $g(p) = 3(p-1)\ln 2 - (p+1)\ln p$ for $p > 0$. Then

$$g'(p) = 3\ln 2 - \ln p - \frac{p+1}{p},$$
$$g''(p) = \frac{1-p}{p^2}.$$

We have $g\left(\frac{1}{2}\right)=0, g(1)=0, g(2)=0$ and $g''(1)=0$. Since $g''(p)>0$ for $0<p<1$ and $g''(p)<0$ for $p>1$, we can deduce that $g(p)<0$ for $\frac{1}{2}<p<1$ and $g(p)>0$ for $1<p<2$. Moreover, $g'(p)<g'\left(\frac{1}{2}\right)=4\ln 2-3<0$ for $0<p<\frac{1}{2}$ and $g'(p)<g'(2)=2\ln 2-\frac{3}{2}<0$ for $p>2$, it follows $g(p)>g\left(\frac{1}{2}\right)$ for $0<p<\frac{1}{2}$ and $g(p)<g(2)$ for $p>2$. Therefore

$$3(p-1)\ln 2-(p+1)\ln p>0 \text{ for } 0<p<\frac{1}{2} \text{ and } 1<p<2,$$

$$3(p-1)\ln 2-(p+1)\ln p<0 \text{ for } \frac{1}{2}<p<1 \text{ and } p>2.$$

Consequently, $\frac{p-1}{\ln p}<\frac{p+1}{3\ln 2}$ for $0<p<\frac{1}{2}$ and $p>2$, $\frac{p-1}{\ln p}>\frac{p+1}{3\ln 2}$ for $\frac{1}{2}<p<1$ and $1<p<2$. Hence $e(p,1)<e\left(\frac{p+1}{3},\frac{2(p+1)}{3}\right)$ for $0<p<\frac{1}{2}$ and $p>2$, $e(p,1)>e\left(\frac{p+1}{3},\frac{2(p+1)}{3}\right)$ for $\frac{1}{2}<p<1$ and $1<p<2$.

Thus the theorem is proved by Lemma 2 and observe that $S_0(x,y)<A_{\frac{1}{3}}(x,y)$ [5, Theorem 1], $S_1(x,y)>A_{\frac{2}{3}}(x,y)$ [1, Theorem 1], $S_{-1}(x,y)=A_0(x,y)$, $S_{\frac{1}{2}}(x,y)=A_{\frac{1}{2}}(x,y)$ and $S_2(x,y)=A_1(x,y)$.

Remark 2 Theorem 4 is just the Theorem 2 of [1]. However, we here give a different and simpler proof.

In[6], Yang and Cao had considered the Heron mean $H_e(x,y)$ and got the inequalities $G(x,y)\le L(x,y)\le H_e(x,y)\le A(x,y)$. Notice that

$$H_e(x,y)=E\left(\frac{1}{2},\frac{3}{2};x,y\right),$$

it is not difficult to find a refinement that

$$G(x,y)\le L(x,y)\le H_e(x,y)\le I(x,y)\le A(x,y).$$

Theorem 5 *We have $A_{\frac{\ln 2}{\ln 3}}(x,y)\le H_e(x,y)\le A_{\frac{2}{3}}(x,y)$ with equality holds if and only if $x=y$.*

Proof Notice that

$$A_{\frac{\ln 2}{\ln 3}}(x,y)=E\left(\frac{\ln 2}{\ln 3},\frac{2\ln 2}{\ln 3};x,y\right) \text{ and } A_{\frac{2}{3}}(x,y)=E\left(\frac{2}{3},\frac{4}{3};x,y\right),$$

we need only to check the condition (3). Since

$$\min\left\{\frac{\ln 2}{\ln 3},\frac{2\ln 2}{\ln 3},\frac{1}{2},\frac{3}{2}\right\}>0 \text{ and } \min\left\{\frac{1}{2},\frac{3}{2},\frac{2}{3},\frac{4}{3}\right\}>0$$

we have

$$e\left(\frac{\ln 2}{\ln 3},\frac{2\ln 2}{\ln 3}\right)=\frac{1}{\ln 3}, e\left(\frac{1}{2},\frac{3}{2}\right)=\frac{1}{\ln 3}, e\left(\frac{2}{3},\frac{4}{3}\right)=\frac{2}{3\ln 2}.$$

Hence

$$e\left(\frac{\ln 2}{\ln 3}, \frac{2\ln 2}{\ln 3}\right) = e\left(\frac{1}{2}, \frac{3}{2}\right) < e\left(\frac{2}{3}, \frac{4}{3}\right).$$

Thus the theorem is proved by Lemma 2.

Corollary *We have $H_e(x,y) \le I(x,y)$ with equality holds if and only if $x = y$.*

References:

[1] STOLARSKY K B. *The power and generalized logarithmic means* [J]. Amer. Math. Monthly, 1980, **87**: 545–548.

[2] VAMANAMURTHY M K, VUORINEN M. *Inequalities for means* [J]. J. Math. Anal. Appl., 1994, **183**: 155–166.

[3] LEACH E B, SHOLANDER M C. *Extended mean values* [J]. Amer. Math. Monthly, 1978, **85**: 84–90.

[4] PÁLES Z. *Inequalities for differences of powers* [J]. J. Math. Anal. Appl., 1988, **131**: 271–281.

[5] LIN T P. *The power mean and the logarithmic mean* [J]. Amer. Math. Monthly, 1974, **81**: 879–883.

[6] YANG Ren-er, CAO Dong-ji. *Generalizations of the logarithmic mean* [J]. J. Ningbo Univ., 1989, **2**: 105–108.

一些平均值的比较

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摘 要: 利用关于广义平均值进行比较的已知结果去比较幂平均值、Stolarsky 平均值和 Heron 平均值，简化了某些结果的证明，并得到了一些新的结果.