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Solution and Coupled Minimal-Maximal Quasi-Solutions of Nonlinear Non-monotone Operator Equations in Banach Spaces *

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Abstract: In this paper, we discuss the existence of the solution and coupled minimal and maximal quasi-solutions for nonlinear non-monotone operator equation $x = A(x, x)$, improved and generalized many relevant results.

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1. Introduction and preliminaries

In this paper, we discuss the solution of the following operator equation:

$$x = A(x, x) \tag{1.1}$$

under the condition that "$A(x, y) + Tx$" is a mixed monotone operator.

Let (E, P) is an ordered Banach space, the norm in $E \times E$ is defined by $\|(x, y)\|_{E\times E} = \max\{\|x\|, \|y\|\}, (x, y) \in E \times E$, then $E \times E$ is a Banach space with $\|\cdot\|_{E\times E}$. Let $\tilde{P} = P \times (-P)$. It is to easy that $\tilde{P}$ is a cone in $E \times E$, and $\tilde{P}$ is a total order minihedral cone (please see the definition in [3]) if P is a total minihedral cone.

$D \subset E$, $A: D \times D \longrightarrow E$, A is called semi-continuous in the first variable if for any fixed $y \in D$ and monotone sequence $\{x_n\}$, $x_n \to x$ implies that $A(x_n, y)$ weakly converges to $A(x, y)$. Similarly, we can define the semi-continuity of A in the second variable.

Let $L(E)$ be the space of linear operators on E and $T \in L(E)$. Define $\gamma(T) = \inf\{k \geq 0, \alpha(T(B)) \leq k\alpha(B), B \subset E \text{ is a bounded set}\}$, where α is Kuratowski measure of noncompactness. T is called positive operator if $x \geq \theta$ deduces $Tx \geq \theta$.

2. Main results

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Theorem 2.1 *Assume E is a real Banach space, P is a total order minihedral cone in E. $D = [u_0, v_0] \subset E$, $A: D \times D \to E$ satisfies the following conditions:*

(i) $u_0 \le A(u_0, v_0)$, $A(v_0, u_0) \le v_0$;

(ii) For any fixed $x \in D$, $A(x, y)$ is decreasing with y;

(iii) There exists a bounded linear positive operator $T: E \to E$ such that for any fixed $y \in D$, we have $A(x_2, y) - A(x_1, y) \ge -T(x_2 - x_1), u_0 \le x_1 \le x_2 \le v_0$;

(iv) There is $\lambda \in (0,1]$ such that $(\lambda I + T)^{-1} \in L(E)$ exists and $(\lambda I + T)x \ge \theta \Rightarrow x \in P$.

Then Eq. (1.1) has coupled minimal and maximal quasi-solutions $(\overline{u}, \overline{v}) \in D \times D$.

Proof Set $G(x, y) = (\lambda I + T)^{-1}[\lambda A(x, y) + Tx], x, y \in D$, then it follows from condition (iv) that $(\lambda I + T)^{-1}$ is positive operator (see [1]), which together with condition (i) implies that

$$u_0 \le G(u_0, v_0), \quad G(v_0, u_0) \le v_0. \tag{2.1}$$

By condition (ii)(iii), we can prove that G is a mixed monotone operator, which together with (2.1) deduces $G: D \times D \to D$

(I) Firstly, we shall show that operator equation

$$x = G(x, x) \tag{2.2}$$

has at least one coupled quasi-solutions in $D \times D$. Let $R = \{(x, y) \in D \times D |\ x \le G(x, y), G(y, x) \le y\}$, then $R \ne \emptyset$ since $(u_0, v_0) \in D$. From Zorn's lemma we can conclude that R contains a maximal element $(x^*, y^*) \in D \times D$, which satisfies

$$x^* \le G(x^*, y^*), \quad G(y^*, x^*) \le y^*, \tag{2.3}$$

from (2.3) and the mixed monotoneity of G we have

$$G(x^*, y^*) \le G((x^*, y^*), (y^*, x^*)), \quad G(y^*, x^*) \ge G((y^*, x^*), (x^*, y^*)), \tag{2.4}$$

(2.4) implies that $(G(x^*, y^*), G(y^*, x^*)) \in R$, and by maximality of (x^*, y^*) we have

$$G(x^*, y^*) \le x^*, \quad G(y^*, x^*) \ge y^*, \tag{2.5}$$

it follows from (2.4)(2.5) that $x^* = G(x^*, y^*)$, $G(y^*, x^*) = y^*$, i.e., (x^*, y^*) are the coupled quasi-solutions of operator equation (2.2).

(II) Secondly, we will show that Eq. (2.2) has coupled minimal and maximal quasi—solutions in $D \times D$. Set

$$F(G) = \{(x, y) \in D \times D |\ (x, y) \text{ is the coupled quasi-solutions of Eq. (2.2)}\},$$

$$S = \{[u, v] \subset E \text{ is order interval} |\ u \le G(u, v), G(v, u) \le v, F(G) \subset [u, v] \times [u, v]\}.$$

Since $D \in S$, $S \ne \emptyset$, and by part (I) we know that $F(G) \ne \emptyset$. Define partial order " $\le$ " in S as following: $[u_1, v_1]$, $[u_2, v_2] \in S$, $[u_1, v_1] \le [u_2, v_2]$ if and only if $[u_1, v_1] \subset [u_2, v_2]$. Suppose that $\{I_\alpha = [u_\alpha, v_\alpha] |\ \alpha \in \Lambda\}$ (Λ is index set) is a completely ordered subset of S, set $Q_1 = \{u_\alpha | \alpha \in \Lambda\}$, $Q_2 = \{v_\alpha | \alpha \in \Lambda\}$, $\overline{I} = [\overline{c}, \overline{w}]$, where $\overline{c}$ and $\overline{w}$ are the minimal upper bound and maximal lower bound of Q_1 and Q_2, respectively. It is easy to see that $\overline{I}$ is a lower bound of $\{I_\alpha|\alpha \in \Lambda\}$ in S, it follows therefore from zorn's lemma that S contains minimal element $[\overline{u},\overline{v}]$ and $[\overline{u},\overline{v}] \in F(G)$. From $[\overline{u},\overline{v}] \in S$ we know that $F(G) \subset [\overline{u},\overline{v}] \times [\overline{u},\overline{v}]$, i.e., $(\overline{u},\overline{v})$ are the coupled minimal and maximal quasi-solutions of (1.1).

Theorem 2.2 *Let E is a norm linear space, $P \subset E$ is a positive cone. Suppose that $A: D \times D \to E$ satisfies conditions (i)-(iv) in Theorem 2.1 and the following condition*

(v) Every completely ordering subset in D is relatively compact.

Then the conclusions of Theorem 2.1 hold.

Proof Let $G, R, F(G), S, I_\alpha, Q_1, Q_2$ be the same as those in the proof of Theorem 2.1. From condition (v) and zorn's lemma we can conclude that R has maximal element $(x^*, y^*) \in D \times D$, and similar to the proof of Theorem 2.1, we can prove that (x^*, y^*) are the coupled quasi-solutions of Eq. (2.2), i.e., $F(G) \neq \emptyset$, $S \neq \emptyset$.

For any completely ordering subset $\{I_\alpha|\alpha \in \Lambda\}$ of S, evidently, Q_1, Q_2 are completely ordering subset, hence they are separable from condition (v), so there exist countable dense subsets $\{a_n\}$ and $\{b_n\}$ of Q_1 and Q_2, respectively. Set $c_n = \max\{a_1, a_2, \cdots, a_n\}$, $w_n = \min\{b_1, b_2, \cdots, b_n\}$, then

$$u_0 \le c_1 \le c_2 \le \cdots \le c_n \le \cdots \le v_0, \tag{2.6}$$

$$u_0 \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le v_0. \tag{2.7}$$

From condition (v), there exist subsequences $\{c_{n_k}\}$ and $\{w_{n_k}\}$ of $\{c_n\}$ and $\{w_n\}$, such that

$$c_{n_k} \to \overline{c}, \quad w_{n_k} \to \overline{w}. \tag{2.8}$$

Similar to the proof of Theorem 2.1, we can show that $\overline{I} = [\overline{c}, \overline{w}]$ is a lower bound of $\{I_\alpha|\alpha \in \Lambda\}$ in S. It follows then from zorn's lemma that S contains minimal element $[\overline{u},\overline{v}]$, and $[\overline{u},\overline{v}]$ are the coupled minimal and maximal quasi-solutions of Eq. (1.1).

Theorem 2.3 *Let E is a Banach space and conditions of Theorem 2.1 or Theorem 2.2 be satisfied. Suppose in addition that $A(x,x)$ is continuous in x, and the following conditions*

(vi) $\gamma[(\lambda I + T)^{-1})] \le \dfrac{1}{\lambda + \gamma(T)}$;

(vii) For any countable set $C \subset D$, $\alpha(A(C,C)) < \alpha(C)$

hold. Then Eq. (1.1) has at least a solution w^* satisfying $\overline{u} \le w^* \le \overline{v}$, where $(\overline{u},\overline{v})$ are the coupled minimal and maximal quasi-solutions of (1.1).

Proof Set $Fx = G(x,x)$, $x \in [\overline{u},\overline{v}]$, then F is continuous. For $\forall x \in [\overline{u},\overline{v}]$, by the mixed monotoneity of G, we have $\overline{u} = G(\overline{u},\overline{v}) \le Fx = G(x,x) \le G(\overline{v},\overline{u}) = \overline{v}$, i.e., $F: [\overline{u},\overline{v}] \to [\overline{u},\overline{v}]$. For some $x \in [\overline{u},\overline{v}]$ and countable set $C \subset [\overline{u},\overline{v}]$ and $\overline{C} = \overline{co}(\{x\} \cup F(C))$, from conditions (vi), (vii) we can conclude that C is relatively compact, then by [2] Theorem 2.1 we know that F has a fixed point $w^* \in [\overline{u},\overline{v}]$, i.e., w^* is a solution of (1.1).

Theorem 2.4 *Let E be a real Banach space and P be a positive cone in E. Suppose that $D = [u_0, v_0]$ is bounded according to norm $\|\cdot\|_E$, $A: D \to D$ is semi-continuous in each variable. If the conditions (i)–(iv), (vi) and the following condition* *(viii) For any countable bounded sets $B_1,\ B_2 \subset D$ with $\max\{\alpha(B_1), \alpha(B_2)\} > 0$, we have*

$$\alpha(A(B_1, B_2)) < \max\{\alpha(B_1), \alpha(B_2)\}, \tag{2.9}$$

hold. Then Eq. (1.1) has coupled minimal and maximal quasi-solutions $(\overline{u}, \overline{v}) \in D \times D$, such that

$$\lim_{n\to\infty} u_n = \overline{u}, \quad \lim_{n\to\infty} v_n = \overline{v}, \tag{2.10}$$

where $u_n = (\lambda I+T)^{-1}[\lambda A(u_{n-1}, v_{n-1})+Tu_{n-1}]$, $v_n = (\lambda I+T)^{-1}[\lambda A(v_{n-1}, u_{n-1})+Tv_{n-1}]$, $n = 1, 2, \cdots$, which satisfy

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le \overline{u} \le \overline{v} \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.11}$$

If we further demand that $A(x, x)$ is continuous in x, then (1.1) has at least a solution u^* satisfying $\overline{u} \le u^* \le \overline{v}$.

Proof Let G be the same as that in Theorem 2.1, then G is a mixed monotone operator, which together with condition (i) deduces the following monotone sequence:

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.12}$$

Set $B_1 = \{u_n | n = 1, 2, \cdots\}$, $B_2 = \{v_n | n = 1, 2, \cdots\}$, then from the conditions we know that $B_1,\ B_2$ are relatively compact, therefore there exist subsequences $\{u_{n_k}\} \subset \{u_n\}$ and $\{v_{n_k}\} \subset \{v_n\}$ such that $u_{n_k} \to \overline{u}$, $v_{n_k} \to \overline{v}$. By indirect arguments we can prove that $u_n \to \overline{u}$, $v_n \to \overline{v}$. Furthermore, (2.12) implies (2.11).

It follows from the monotoneity of G and (2.11) that $u_{n+1} = G(u_n, v_n) \le G(\overline{u}, \overline{v}) \le G(\overline{v}, \overline{u}) \le G(v_n, u_n) = v_{n+1}$, $n = 1, 2, \cdots$. Let $n \to \infty$, we obtain

$$\overline{u} \le G(\overline{u}, \overline{v}) \le G(\overline{v}, \overline{u}) \le \overline{v}. \tag{2.13}$$

On the other hand, for any $n, k \in N$,

$$G(u_n, v_{n+k}) \le G(u_{n+k}, v_{n+k}) = u_{n+k+1},\ v_{n+k+1} = G(v_{n+k}, u_{n+k}) \le G(v_n, u_{n+k}), \tag{2.14}$$

then by (2.14) and the semi—continuity of $G(x,)$ and $G(, y)$ we have

$$G(\overline{u}, \overline{v}) \le \overline{u},\ \overline{v} \le G(\overline{v}, \overline{u}). \tag{2.15}$$

It follows then from (2.13)(2.15) that $\overline{u} = G(\overline{u}, \overline{v})$, $G(\overline{v}, \overline{u}) = \overline{v}$, i.e., $(\overline{u}, \overline{v})$ are the coupled quasi-solutions of (2.2). Similar to the proof of [5] Theorem 2.1.2, we can show that $(\overline{u}, \overline{v})$ are the coupled minimal and maximal quasi-solutions of (2.2). Similar to proof of Theorem 2.3, we can obtain a solution of (2.2). Then by the definition of G, we know that the conclusions of Theorem 2.4 hold.

3. Applications

In this section, we will discuss the following nonlinear implusive integral equation:

$$x(t) = \int_0^a g(t, s)H(s, x(s), x(s))\mathrm{d}s + \sum_{0<t_i<t} I_i(x(t_i), x(t_i)), \tag{3.1}$$

where $g(t,s) \in C[J \times J, R^+]$, R^+ is nonnegative real number set, $H \in C[J \times E \times E, E]$,$I_i \in C[E \times E, E]$, $i = 1, 2, \cdots, m$, E is a real Banach space and $J = [0, a]$, $0 < t_1 < t_2 < \cdots < t_i < \cdots < t_m < a$. Let $PC[J, E] = \{x : J \to E$ such that $x(t)$ is continuous at $t \neq t_i$, and left continuous at $t = t_i$, and the right limit $x(t_i + 0) = \lim_{t \to t_i^+} x(t)$ exists for $i = 1, 2, \cdots, P\}$. Evidently, $PC[J, E]$ is a Banach space with norm: $\|x\|_{PC} = \sup_{t \in J} \|x\|$. we always denote $[u_0, v_0]_{PC} = \{u \in PC[J, E] : u_0 \le u \le v_0\}$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_m = (t_m, a]$.

(A$_1$) There exist u_0, $v_0 \in PC[J, E]$ such that $[u_0, v_0]$ is bounded according to norm $\|\cdot\|_E$, and satisfy

$$u_0(t) \le \int_0^a g(t,s)H(s, u_0(s), v_0(s))\mathrm{d}s + \sum_{0<t_i<t} I_i(u_0(t_i), v_0(t_i)), \ t \in J,$$

$$v_0(t) \ge \int_0^a g(t,s)H(s, v_0(s), u_0(s))\mathrm{d}s + \sum_{0<t_i<t} I_i(v_0(t_i), u_0(t_i)), \ t \in J;$$

(A$_2$) $I_i(x, y)$ is increasing in x and decreasing in y, $i = 1, 2, \cdots, m$;

(A$_3$) There exists positive continuous function $f(t)$, such that for any $x_1, x_2, y_1, y_2 \in [u_0, v_0]_{PC}$ with $x_1 \le x_2$, $y_1 \le y_2$, we have

$$\int_0^a g(t,s)H(s, x_2(s), y_1(s))\mathrm{d}s - \int_0^a g(t,s)H(s, x_1(s), y_2(s))\mathrm{d}s \ge -f(t)(x_2(t) - x_1(t)),$$

and for fixed $x \in [u_0, v_0]_{PC}$, $H(t, x, y)$ is decreasing in y;

(A$_4$) There are $k, l_i \in C[J, R^+]$ $(i = 1, 2, \cdots, m)$ satisfying $2\int_0^a g(t,s)k(s)\mathrm{d}s + \sum_{i=1}^m l_i(t) < 1$, such that for any countable bounded sets B_1, B_2 and $t \in J$, we have

$$\alpha(H(t, B_1, B_2)) \le k(t)\max(\alpha(B_1), \alpha(B_2)),$$
$$\alpha(I_i(B_1, B_2)) \le l_i(t)\max(\alpha(B_1), \alpha(B_2)), \quad i = 1, 2, \cdots, m.$$

Theorem 3.1 *Let E be a Banach space and P be a cone in E. Suppose that conditions (A$_1$)—(A$_4$) hold, then Eq. (3.1) must have a solution w^* and coupled minimal and maximal quasi-solutions $(\overline{u}, \overline{v})$, which satisfy $u_0 \le \overline{u} \le w^* \le \overline{v} \le v_0$. Furthermore, there exist $\{u_n\}$, $\{v_n\} \subset [u_0, v_0]_{PC}$ such that $u_n \to \overline{u}$, $v_n \to \overline{v}$ and satisfy*

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le \overline{u} \le w^* \le \overline{v} \le \cdots \le v_n \le \cdots \le v_1 \le v_0.$$

Proof Set

$$A(x, y) = \int_0^a g(t,s)H(s, x(s), y(s))\mathrm{d}s + \sum_{0<t_i<t} I_i(x(t_i), y(t_i)). \tag{3.2}$$

It is easy to prove that A is continuous. By (A$_1$) we know $u_0 \le A(u_0, v_0)$, $A(v_0, u_0) \le v_0$. It follows from (A$_2$), (A$_3$) that for fixed $x \in D = [u_0, v_0]_{PC}$, $A(x, y)$ is decreasing in y and for fixed $y \in D$, $\forall x_1, x_2 \in D$ with $x_1 \le x_2$ we have

$$A(x_2, y) - A(x_1, y) \ge -f(t)(x_2(t) - x_1(t)).$$

Define $Tx(t) = f(t)x(t)$, then $T \in L(E)$, and $(\lambda I + T)^{-1} = \frac{1}{\lambda + f(t)}x(t)$, $t \in J$. For any bounded set $B \subset [u_0, v_0]_{PC}$, we have $\alpha(T(B)) = f(t)\alpha(B)$, then we can obtain $\gamma[(\lambda I + T)^{-1}(B)] = \frac{1}{\lambda + \max\limits_{t\in J} f(t)}$. Since $\alpha(T(B)) = f(t)\alpha(B)$, $\gamma(T) = \min\limits_{t\in J} f(t)$, we get $\gamma[(\lambda I+T)^{-1}(B)] = \frac{1}{\lambda+\max\limits_{t\in J} f(t)} \leq \frac{1}{\lambda+\min\limits_{t\in J} f(t)} = \frac{1}{\lambda+\gamma(T)}$, i.e., condition (vi) is verified. For any countable bounded sets $B_1 = \{x_n\}$, $B_2 = \{y_n\} \subset [u_0, v_0]_{PC}$, it follows from (3.2), (A_2) and (A_3) that $A(B_1, B_2)$ is bounded. Since $g(t,s)$ is continuous, $A(B_1, B_2)$ is equicontinuous. From [4] Lemma 3 we get

$$\alpha(A(B_1, B_2)) = \sup_{t\in J} \alpha(A(B_1(t), B_2(t))). \tag{3.3}$$

For each n, because $x_n(t)$, $y_n(t)$ are continuous in $t \in J_i, (i = 1, 2, \cdots, m)$, hence $\{x_n(t)|t \in J\} \cup \{y_n(t)|t \in J\}$ is a separable set in E, so we have $\{x_n(t)|t \in J, n \in N\} \cup \{y_n(t)|t \in J, n \in N\}$ is a separable set in E, then without loss of generality, we can suppose that E is a separable Banach space. Thus by (3.2), (A_4) and [4] Lemma 4 we obtain

$$\alpha(A(B_1(t), B_2(t))) \leq \max\{\alpha(B_1), \alpha(B_2)\}. \tag{3.4}$$

It follows from (3.3) and (3.4) that (viii) holds, then the conclusion follows from Theorem 2.4.

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Banach 空间非线性非单调算子方程的解和最小 - 最大拟解对

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摘 要: 本文讨论了非线性非单调算子方程 $x = A(x, x)$ 的解和最小 - 最大拟解对的存在性, 改进并推广了若干结果.