Journal of Mathematical Research & Exposition
Vol.23, No.1, 65-70, February, 2003

Some Remarks on Rational Interpolation to $|x|$ *

ZHAO Yi[1,2], *ZHOU Song-ping*[2]
(1. Inst. of Math., Ningbo University, Zhejiang 315211, China;
2. Dept. of Math., Zhejiang University, Hangzhou 310028, China)

Abstract: The present paper constructs a set of nodes which can generate a rational interpolating function to approximate $|x|$ at the rate of $O(1/(n^k \log n))$ for any given natural number k. More importantly, this construction reveals the fact that the higher density the distribution of a set of nodes has to zero (that is the singular point of the function $|x|$!), the better the rational interpolation approximation behaves. This probably also provides an idea to construct more valuable sets of nodes in the future.

Key words: rational approximation; interpolation; nodes.

Classification: AMS(2000) 41A20,41A05/CLC number: O174.41

Document code: A **Article ID:** 1000-341X(2003)01-0065-06

1. Introduction

In approximation Theory, the function $|x|$ always plays a very important role. It is well known that the best uniform approximation to $|x|$ by *polynomials of degree n* can achieve the rate $O(n^{-1})$, and this rate cannot be improved further (cf. Bernstein [1]). However, in 1964, Newman [4] proved that *rational approximation* to $|x|$ has much more benefits, exactly, $|x|$ can be approximated uniformly by rational functions at a nearly exponential rate. It must be pointed out that the rational function Newman used in his proof interpolates $|x|$ at the nodes $\{-a, -a^2, \cdots, -a^{n-1}, 0, a^{n-1}, \cdots, a^2, a\}$, $a = e^{1/\sqrt{n}}$.

Recently, following Werner[5], Bruteman and Passow[3] and Bruteman[2] consider Newman type approximation induced by arbitrary sets of interpolation points. Let[1] $X = \{0 < x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \le 1\}$, $p(x) = \prod_{k=1}^n (x + x_k^{(n)})$, and the rational function corresponding to the set X is defined by

$$r_n(X; x) = x \frac{p(x) - p(-x)}{p(x) + p(-x)}. \tag{1}$$

*Received date: 2000-05-28
Foundation item: The Research Project of Institute of Mathematics, Ningbo University. Supported in part by National and Provincial Natural Science Foundations (under grant numbers 10141001 and 101009), and by Ningbo Key Doctoral Funds.
Biography: ZHAO Yi (1976-), female, Ph.D.

[1] In the sequel, the superscript (n) will be omitted if there is no possibility of confusion.

For this function $r_n(X;x)$, we can easily verify that it interpolates $|x|$ at the set of $2n+1$ points $\{-x_n,\cdots,-x_1,0,x_1,\cdots,x_n\}$.

Werner first investigated the equidistant nodes and proved that the exact rational interpolating approximation rate to $|x|$ is $O(1/(n\log n))$. As people often think that the roots of Chebyshev polynomials (Chebyshev nodes) play an optimal role in polynomial interpolation problems, Chebyshev nodes are also studied by Bruteman and Passow[3] accordingly. However, they showed a disappointed result as well as these nodes can only achieve the exact approximation rate $O(1/(n\log n))$ to $|x|$, like the equidistant nodes.

Bruteman[2] shifts attention to the set of nodes obtained by adjusting the Chebyshev roots $\xi_k^{(n)}=\cos((2k-1)\pi/(2n))$, $k=1,2,\cdots,n$ to the interval $[0,1]$, namely,

$$X=\left\{x_k=\frac{1}{2}(1+\xi_{n-k+1}^{(n)})=\sin^2((2k-1)\pi/(4n))\right\}_{k=1}^{n},$$

and proved that the exact order of the rational interpolating approximation to $|x|$ is $O(1/n^2)$ in this case.

These three sets of nodes lead people to ask a natural question why they behave like this, more exactly, we have the following questions: Why do the optimal nodes in polynomial interpolation behave so "badly" in rational interpolation case? Why can the above mentioned "adjusted" Chebyshev nodes get better order? What kind of nodes can be suitable for rational interpolation purpose?

The present paper constructs a set of nodes which can generate a rational interpolating function to approximate $|x|$ at the rate of $O(1/(n^k\log n))$ for any given natural number k. More importantly, this construction reveals the fact that if the distribution of a set of nodes has higher density to zero (that is the singular point of the function $|x|$!), then the rational interpolation approximation behaves better. That is why the Chebyshev nodes behaves "badly" in rational interpolation case (since its density concentrates to the endpoints), and why the "adjusted" Chebyshev nodes behaves better (since its density concentrates to zero and ±1). This probably also provides an idea to construct more valuable sets of nodes in the future.

2. Results

In this section, we first construct a particular set of nodes to illustrate the idea, and the general case will be considered similarly.

Suppose n is an odd number, say $n=2m-1$, and set

$$x_1=1/m^2,\ \ x_2=2/m^2,\ \ \cdots,\ \ x_{m-1}=(m-1)/m^2,$$

$$x_m=1/m,\ \ x_{m+1}=2/m,\ \ \cdots,\ \ x_{2m-2}=(m-1)/m,\ \ x_{2m-1}=1.$$

In the case n is an even number, say $n=2m$, and set

$$x_1=1/(m+1)^2,\ \ x_2=2/(m+1)^2,\ \ \cdots,\ \ x_m=m/(m+1)^2,$$

$$x_{m+1}=1/(m+1),\ \ x_{m+2}=2/(m+1),\ \ \cdots,\ \ x_{2m}=m/(m+1).$$

Then the rational function $r_n(X;x)$ corresponding to the set $X = \{x_k\}_{k=1}^n$ as defined in (1) interpolates $|x|$ at the points $\{-x_n, -x_{n-1}, \cdots, -x_1, 0, x_1, \cdots, x_{n-1}, x_n\}$. We have

Theorem 1 *For sufficiently large n, the estimate $|x| - r_n(X;x) = O\left(\frac{1}{n^2 \log n}\right)$ holds. Moreover, the approximation order is exact.*

We only prove Theorem 1 for the case when n is an odd number, and the other can be treated similarly. We need to establish the following Lemma.

Lemma *Let n be an odd number, and*

$$h_n(X;x) = \frac{p(-x)}{p(x)} = \prod_{k=1}^{n} \frac{x_k - x}{x_k + x}, \tag{2}$$

then for $x \in [1/m^2, 1]$ and sufficiently large n, $h_n(X;x) = O(1/n)$.

Proof First let $x \in [j/m^2, (j+1)/m^2]$, $j = 1, 2, \cdots, m-1$. Then

$$\begin{aligned}|h_n(X;x)| &= \prod_{k=1}^{j} \frac{x - k/m^2}{x + k/m^2} \prod_{k=j+1}^{m} \frac{k/m^2 - x}{k/m^2 + x} \prod_{k=2}^{m} \frac{k/m - x}{k/m + x} \\ &\le \prod_{k=1}^{j} \frac{x - k/m^2}{x + k/m^2} \prod_{k=j+1}^{m} \frac{k/m^2 - x}{k/m^2 + x}.\end{aligned}$$

From the well-known inequality $1 - x \le e^{-x}$ for $x \ge 0$,

$$|h_n(X;x)| \le \exp\left(-x^{-1}m^{-2}\sum_{k=1}^{j} k\right) \exp\left(-xm^2 \sum_{k=j+1}^{m} k^{-1}\right) = O(e^{-j/2}e^{-j\log(m/j)}). \tag{3}$$

It is easy to calculate that $e^{-j/2}e^{-j\log(m/j)} = O(1/m) = (1/n)$ for $1 \le j \le m-1$, so our lemma is proved for $x \in [1/m^2, 1/m]$. Now suppose that $x \in [1/m, 1]$, say $x \in [j/m, (j+1)/m]$, $j = 1, 2, \cdots, m-1$, similar to the above case,

$$\begin{aligned}|h_n(X;x)| &\le \prod_{k=1}^{j} \frac{x - k/m}{x + k/m} \prod_{k=j+1}^{m} \frac{k/m - x}{k/m + x} \\ &\le \exp\left(-x^{-1}m^{-1}\sum_{k=1}^{j} k\right) \exp\left(-xm \sum_{k=j+1}^{m} k^{-1}\right) = O(1/n),\end{aligned}$$

thus we completed the proof of the lemma. □

Proof of Theorem 1 Without loss of generality, we only need to prove Theorem 1 when n is an odd number. Since both $|x|$ and $r_n(X;x)$ are even functions, we can also only consider the approximation on the interval $[0,1]$. Then

$$x - r_n(X;x) = \frac{2xh_n(X;x)}{1 + h_n(X;x)}, \quad x \in [0,1],$$

where $h_n(X;x)$ is defined in (2). The proof will be divided into the following cases.

Case 1 $x \in [0, 1/m^2]$. Since $h_n(X;x) \geq 0$ in this case, we see

$$|x - r_n(X;x)| = \frac{2xh_n(X;x)}{1 + h_n(X;x)} \leq 2xh_n(X;x),$$

$$h_n(X;x) = \prod_{k=1}^{m} \frac{k/m^2 - x}{k/m^2 + x} \prod_{k=2}^{m} \frac{k/m - x}{k/m + x} \leq \prod_{k=1}^{m} \frac{k/m^2 - x}{k/m^2 + x}$$

$$\leq \prod_{k=1}^{m} \left(1 - \frac{xm^2}{k}\right) \leq \exp\left(-xm^2 \sum_{k=1}^{m} k^{-1}\right) \leq \exp(-xm^2 \log m).$$

It is clear that the function $x \exp(-xm^2 \log m)$ achieves its maximum value in the interval $[0, 1/m^2]$ at $x = 1/(m^2 \log m)$, so that

$$|x - r_n(X;x)| \leq 2xh_n(X;x) \leq 2x \prod_{k=1}^{m} \frac{k/m^2 - x}{k/m^2 + x} \leq \frac{2e^{-1}}{m^2 \log m}, \tag{4}$$

that is the required result.

Case 2 $x \in [1/m^2, 1/(2m)]$. Applying the lemma, for sufficiently large n we get

$$|x - r_n(X;x)| = \frac{2x|h_n(X;x)|}{|1 + h_n(X;x)|} = O(x|h_n(X;x)|).$$

Noting that

$$x|h_n(X;x)| \leq x \prod_{k=1}^{m} \left|\frac{k/m^2 - x}{k/m^2 + x}\right|, \tag{5}$$

by setting $g(x) = x \prod_{k=1}^{m} \left|\frac{k/m^2 - x}{k/m^2 + x}\right|$, we calculate

$$\frac{g(x)}{g(x - 1/m^2)} = \frac{x}{x - 1/m^2} \prod_{k=1}^{m} \left|\frac{k/m^2 - x}{k/m^2 + x} \frac{(k-1)/m^2 + x}{(k+1)/m^2 - x}\right|$$

$$= \frac{x}{x - 1/m^2} \frac{x - 1/m^2}{(m+1)/m^2 - x} \frac{x}{1/m + x} = \frac{x^2}{1/m + x} \frac{1}{(m+1)/m^2 - x}$$

$$\leq \frac{x^2}{1/m^2 - x^2}.$$

In view of $x \in [1/m^2, 1/(2m)]$ in this case, from the above inequality we obtain that

$$\frac{g(x)}{g(x - 1/m^2)} \leq \frac{1}{3},$$

with (4), (5), we have

$$|x - r_n(X;x)| = O(x|h_n(X;x)|) = O\left(\max_{0 \leq x \leq 1/(2m)} g(x)\right)$$

$$= O\left(\max_{0 \leq x \leq 1/m^2} g(x)\right) = O(1/(n^2 \log n)).$$

Case 3 $x \in [1/(2m), 1/\sqrt{m}]$. Similar to the proof of the lemma, we immediately achieve that (see the proof of (3))

$$
\begin{aligned}
|x - r_n(X;x)| =& O(|h_n(X;x)|) = O\left(\prod_{k=1}^{[m/2]} \frac{x - k/m^2}{x + k/m^2}\right) \\
=& O\left(\exp\left(-x^{-1}m^{-2}\sum_{k=1}^{[m/2]} k\right)\right) = O(e^{-\sqrt{m}/8}),
\end{aligned}
$$

that also finishes the proof.

Case 4 $x \in [1/\sqrt{m}, 1]$. In this case, we simply calculate that (cf.(3))

$$
|x - r_n(X;x)| = O(|h_n(X;x)|) = O\left(\prod_{k=1}^{[\sqrt{m}]} \frac{x - k/m}{x + k/m}\right) = O\left(e^{-\sqrt{m}/2}\right),
$$

thus the required estimate also holds.

Finally, we need to show that the above estimate cannot be improved. Take $x^* = x^*(n) = 1/(m^2 \log m)$, $x^* \in (0, 1/m^2)$, by the lemma, for sufficiently large n,

$$
|x - r_n(X;x^*)|/x^* = \frac{2h_n(X;x^*)}{1 + h_n(x^*)} \geq h_n(X, x^*). \tag{6}
$$

We see

$$
\begin{aligned}
1 < \frac{1}{h_n(X;x^*)} &= \prod_{k=1}^{m} \frac{k/m^2 + 1/(m^2 \log m)}{k/m^2 - 1/(m^2 \log m)} \prod_{k=2}^{m} \frac{k/m + 1/(m^2 \log m)}{k/m - 1/(m^2 \log m)} \\
&= \prod_{k=1}^{m}\left(1 + \frac{2}{k \log m - 1}\right) \prod_{k=2}^{m}\left(1 + \frac{2}{km \log m - 1}\right) \\
&\leq \exp\left(\sum_{k=1}^{m} 2/(k \log m - 1)\right) \exp\left(\sum_{k=2}^{m} 2/(km \log m - 1)\right) = O(1),
\end{aligned}
$$

together with (6), $|x - r_n(X;x^*)| \geq x^* h_n(X, x^*) \geq \frac{C}{n^2 \log n}$, where $C > 0$ is a positive constant independent of n. Up to this stage, we have finally finished Theorem 1. □

Now we consider the general case. Let k be a given natural number, $km < n \leq (k+1)m$, $m = 0, 1, \cdots$. Since we only investigate the approximation rate for sufficiently large n, without loss of generality assume $m > k + 1$. Write $n = km + j$, and assume that $pk < j \leq (p+1)k$, $p = 0, 1, \cdots$, set

$$
x_1 = \frac{1}{(m+p+2)^k}, \quad x_2 = \frac{2}{(m+p+2)^k}, \quad \cdots, \quad x_{m+p+2} = \frac{1}{(m+p+2)^{k-1}},
$$

$$
x_{m+p+3} = \frac{2}{(m+p+2)^{k-1}}, \quad \cdots, \quad x_{2(m+p+2)-1} = \frac{1}{(m+p+2)^{k-2}}, \quad \cdots,
$$

$$x_{(k-1)(m+p+2)-k+2} = \frac{1}{m+p+2}, \quad x_{(k-1)(m+p+2)-k+3} = \frac{2}{m+p+2}, \quad \cdots,$$

$$x_n = x_{km+j} = \frac{m+j-(p+1)k+p+1}{m+p+2},$$

where we should note that $2 < m-k+p+1 < m+j-(p+1)k+p+1 \le m+p+1$ in view of that $m > k+1$ and $pk < j \le (p+1)k$, $p = 0, 1, \cdots$, thus the index range of the nodes is reasonable. Then, the rational function $r_n(X;x)$ corresponding to the set $X = \{x_k\}_{k=1}^{n}$ as defined in (1) interpolates $|x|$ at the points $\{-x_n, -x_{n-1}, \cdots, -x_1, 0, x_1, \cdots, x_{n-1}, x_n\}$. We have

Theorem 2 *For sufficiently large n, the estimate $||x| - r_n(X;x)| \le \frac{C_k}{n^k \log n}$ holds, where $C_k > 0$ is a positive constant only depending upon k. Moreover, the approximation order is exact.*

The proof is similar to but more complicated than that of Theorem 1, we omit it.

References:

[1] BERNSTEIN S N. *Sur la meilleure approximation de $|x|$ par des polynomes de degrés donnés* [J]. Acta. Math., 1913, **37**: 1–57.

[2] BRUTEMAN L. *On rational interpolation to $|x|$ at the adjusted Chebyshev nodes* [J]. J. Approx. Theory, 1998, **95**: 146–152.

[3] BRUTEMAN L, PASSOW E. *Rational interpolation to $|x|$ at the Chebyshev nodes* [J]. Bull. Austral. Math. Soc., 1997, **56**: 81–86.

[4] NEWMAN D. *Rational approximation to $|x|$* [J]. Michigan Math. J., 1964, **11**: 11–14.

[5] WERNER H. *Rationale Interpolation von $|x|$ in äquidistanten Punkten* [J]. Math. Z., 1982, **180**: 11–17.

$|x|$ 的有理插值的若干注记

赵 易[1,2], 周 颂 平[2]

(1. 宁波大学数学研究所, 浙江 宁波 315211; 2. 浙江大学数学系, 浙江 杭州 310028)

摘 要: 本文中我们构造了一个结点组，基于它定义的有理插值函数，对于任意给定的自然数 k, 对 $|x|$ 的逼近能达到精确阶 $O(1/(n^k \log n))$. 更重要的是，这样的构造揭示了一个本质：当结点向 ($|x|$ 的唯一奇异点) 零点集中时， $|x|$ 的有理插值逼近阶也随之更佳. 这或许为将来本质性的自然结点组的构造提供了一种思路.