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A Note on a Class of Bidimensional Nonseparable Refinable Distributions *

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Abstract: Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. In this paper, an optimal upper bound estimate of the modules of Fourier transforms of M-refinable distributions is obtained by the introduction of cycle related to M.

Key words: cycle; nonseparable refinable distribution.

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1. Introduction

Since the introduction by Daubechies[1] of compactly supported orthogonal wavelet bases in R^1 with arbitrarily high smoothness, various new wavelet bases have been constructed and applied successfully in image processing, numerical computation, statistics, etc. Many of these applications, such as image compression, employ separable wavelet bases in R^2, which are simply tensor products of univariate basis functions. This preferred directions effect can create unpleasant artifacts that become obvious at high image compression ratios. Nonseparable wavelet bases offer the hope of a more isotropic analysis ([2]-[9]).

Throughout this paper, M is always referred to be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. In this note, an optimal upper bound estimate of the modules of Fourier transforms of M- refinable distributions is obtained in terms of cycle related to M.

Definition 1.1 *A finite set* $\{\xi_1, \xi_2, \cdots, \xi_n\} \subset [-\pi, \pi]^2$ *is called a cycle related to* $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ *if* $\xi_{i+1} = \tau\xi_i$, $\xi_1 = \tau\xi_n$, *where* $\tau\xi = M\xi(\mathrm{mod}2\pi Z^2)$ *for* $\xi \in [-\pi, \pi]^2$, τ *maps*

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$[-\pi, \pi]^2$ *into* $[-\pi, \pi]^2$. *A cycle* $\{\xi_1, \xi_2, \cdots, \xi_n\} \subset [-\pi, \pi]^2$ *is called non-trivial if it is different from* $\{0\}$.

Our main results can be stated as follows.

Theorem 1.1 *Let* $\mathcal{L}$ *be a trigonometric polynomial with* $\mathcal{L}(0) = 1$. *Assume that* $[-\pi, \pi]^2 = D_1 \cup D_2 \cup \cdots \cup D_{n_0}$, *and that there exists* $q > 0$ *such that*

$$\begin{array}{ll} |\mathcal{L}(\xi)| \le q, & \xi \in D_1, \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)| \le q^2, & \xi \in D_2, \\ \vdots & \\ |\mathcal{L}(\xi)\mathcal{L}(M\xi)\cdots\mathcal{L}(M^{n_0-1}\xi)| \le q^{n_0}, & \xi \in D_{n_0}. \end{array}$$

Then $|\prod_{j=1}^{\infty} \mathcal{L}(M^{-j}\cdot)| \le C(1+|\cdot|)^{\mathcal{K}}$ *for* $\mathcal{K} = 2\log_2 q$.

Theorem 1.2 *Let* $H_0(\xi) = m_0(\xi_1) = \left(\frac{1+e^{-i\xi_1}}{2}\right)^N \mathcal{L}(\xi_1)$ *for* $\xi = (\xi_1, \xi_2)^T$, m_0 *be a univariate polynomial with* $m_0(0) = 1$. *Assume that* $\{\xi^{(0)}, \xi^{(1)}, \cdots, \xi^{(n_0-1)}\} \subset [-\pi, \pi]^2$ *is a non-trivial cycle related to* M *with* $\xi_1^{(0)} \ne \pm\pi, 0$ *and* $\xi_1^{(0)} + \xi_2^{(0)} \ne 0$, *where* $\xi^{(0)} = (\xi_1^{(0)}, \xi_2^{(0)})^T$. *Define* ϕ *by*

$$\hat{\phi}(\cdot) = \prod_{j=1}^{\infty} H_0(M^{-j}\cdot).$$

Then there exists a constant C *such that*

$$|\hat{\phi}(2^{kn_0} M\xi^{(0)})| \ge C(1+|2^{kn_0}\xi^{(0)}|)^{-2N+\tilde{\mathcal{K}}}$$

for $k \in N$, *where* $\tilde{\mathcal{K}} = \frac{2}{n_0}\sum_{m=0}^{n_0-1} \log_2 |\mathcal{L}(\xi^{(m)})|$.

Remark 1.1 Theorem 1.1 and Theorem 1.2 are optimal. This can be explained in Example 2.1 of Section 2.

2. Proofs of theorems

Proof of Theorem 1.1 For $j > n_0$, any $\xi \in [-\pi, \pi]^2$, there exists $1 \le m \le n_0$ such that $\xi \in D_m$. Hence,

$$|\prod_{k=0}^{j-1} \mathcal{L}(\tau^k\xi)| \le q^m |\prod_{k=m}^{j-1} \mathcal{L}(\tau^k\xi)|.$$

Suppose $\tau^m\xi \in D_{m_1}$, then

$$|\prod_{k=0}^{j-1} \mathcal{L}(\tau^k\xi)| \le q^{m+m_1} |\prod_{k=m+m_1}^{j-1} \mathcal{L}(\tau^k\xi)|.$$

Applying the same trick to $\tau^{m+m_1}\xi$, and keeping doing so until we can not go on, we obtain that

$$|\prod_{k=0}^{j-1} \mathcal{L}(\tau^k\xi)| \le q^{j-r} |\prod_{k=j-1-r}^{j-1} \mathcal{L}(\tau^k\xi)|$$

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with at most $n_0 - 1$ different $\mathcal{L}$-factors remaining. Hence,

$$|\prod_{k=0}^{j-1} \mathcal{L}(\tau^k \xi)| \le C_1 q^j,$$

where $C_1 = \max_{0 \le r < n_0} q^{-r} \max_\xi |\prod_{k=0}^{r} \mathcal{L}(\tau^k \xi)|$.

For $|\xi| > (\sqrt{2})^{n_0}$, there exists $J > n_0$ such that $(\sqrt{2})^{J-1} < |\xi| \le (\sqrt{2})^J$. Hence,

$$\begin{aligned} |\prod_{k=1}^{\infty} \mathcal{L}(M^{-k}\xi)| &= |\prod_{k=1}^{J} \mathcal{L}(M^{-k}\xi)||\prod_{k=J+1}^{\infty} \mathcal{L}(M^{-k}\xi)| \\ &\le C_2 |\prod_{k=1}^{J} \mathcal{L}(M^{-k}\xi)| \le C_3 q^J \le C_4 |\xi|^{2\log_2 q}. \end{aligned}$$

Therefore, $|\prod_{j=1}^{\infty} \mathcal{L}(M^{-j}\xi)| \le C(1+|\xi|)^{2\log_2 q}$ for $\xi \in R^2$. The proof is completed.

Proof of Theorem 1.2

$$\begin{aligned} \hat{\phi}(\xi) &= |\prod_{j=1}^{\infty} H_0(M^{-2j}\xi)||\prod_{j=1}^{\infty} H_0(M^{-2j+1}\xi)| \\ &= |\prod_{j=1}^{\infty} H_0(2^{-j}\xi)||\prod_{j=1}^{\infty} H_0(2^{-j}M\xi)| \\ &= |\frac{\sin\frac{\xi_1}{2}}{\frac{\xi_1}{2}}|^N |\frac{\sin\frac{\xi_1+\xi_2}{2}}{\frac{\xi_1+\xi_2}{2}}|^N |\prod_{j=1}^{\infty} \mathcal{L}(M^{-j}\xi)|, \end{aligned}$$

$$\hat{\phi}(2^{kn_0} M\xi^{(0)}) = |\frac{\sin 2^{kn_0-1}(\xi_1^{(0)}+\xi_2^{(0)})}{2^{kn_0-1}(\xi_1^{(0)}+\xi_2^{(0)})}|^N |\frac{\sin 2^{kn_0}\xi_1^{(0)}}{2^{kn_0}\xi_1^{(0)}}|^N |\prod_{j=1}^{\infty} \mathcal{L}(M^{-j}2^{kn_0}M\xi^{(0)})|.$$

Since

$$M^{2kn_0-1}\xi^{(0)} = M^{-1}(M^{2kn_0}\xi^{(0)}) = M^{-1}[\xi^{(0)}(\mathrm{mod}(2\pi Z^2))] = M^{-1}\xi^{(0)}(\mathrm{mod}(\pi M Z^2)),$$

we have that

$$|\sin 2^{kn_0-1}(\xi_1^{(0)}+\xi_2^{(0)})| = |\sin\frac{\xi_1^{(0)}+\xi_2^{(0)}}{2}|.$$

Since $M^{2kn_0}\xi^{(0)} = \xi^{(0)}(mod(2\pi Z^2))$, $|\sin 2^{kn_0}\xi_1^{(0)}| = |\sin\xi_1^{(0)}|$. These together with the fact that $\xi_1^{(0)} \ne \pm\pi$, 0 and $\xi_1^{(0)}+\xi_2^{(0)} \ne 0$ lead to

$$|\hat{\phi}(2^{kn_0}M\xi^{(0)})| \ge C_1(2^{kn_0}|\xi^{(0)}|)^{-2N} |\prod_{j=1}^{\infty} \mathcal{L}(M^{-j}2^{kn_0}M\xi^{(0)})|. \tag{2.1}$$

Since the trigonometric polynomial $\mathcal{L}$ satisfies $\mathcal{L}(0) = 1$, there exists a constant C_2 such that $|\mathcal{L}(\xi)| \ge 1 - C_2|\xi| \ge e^{-2C_2|\xi|}$ for $|\xi|$ small enough. Hence, there exists a constant C_3

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such that

$$\prod_{j=2rn_0}^{\infty} |\mathcal{L}(M^{-j}\xi^{(0)}) \geq \prod_{j=2rn_0}^{\infty} e^{-2C_2|M^{-j}\xi^{(0)}|}$$

$$= \prod_{j=rn_0}^{\infty} e^{-2C_2 2^{-j}|\xi^{(0)}|} \prod_{j=rn_0}^{\infty} e^{-2C_2 2^{-j}|M^{-1}\xi^{(0)}|}$$

$$= \prod_{j=rn_0}^{\infty} e^{-(2+\sqrt{2})C_2 2^{-j}|\xi^{(0)}|} \geq C_3$$

for r large enough.

This together with (2.1) leads to that

$$|\hat{\phi}(2^{kn_0}M\xi^{(0)})| \geq C_4(2^{kn_0}|\xi^{(0)}|)^{-2N}|\prod_{l=0}^{(2r+2k)n_0-1} \mathcal{L}(M^{2kn_0-l}\xi^{(0)})|$$

$$\geq C_4(1+2^{kn_0}|\xi^{(0)}|)^{-2N}|\mathcal{L}(\xi^{(0)})\mathcal{L}(\xi^{(1)})\cdots\mathcal{L}(\xi^{(n_0-1)})|^{2r+2k}$$

$$\geq C_4(1+2^{kn_0}|\xi^{(0)}|)^{-2N}2^{kn_0\tilde{\mathcal{K}}} \geq C(1+2^{kn_0}|\xi^{(0)}|)^{-2N+\tilde{\mathcal{K}}}.$$

The proof is completed.

Example 2.1 Assume that $H_0(\xi) = \left(\frac{1+e^{-i\xi_1}}{2}\right)^N A(\xi)$ for $\xi = (\xi_1, \xi_2)^T$, and that

$$|A(\xi)|^2 = P_N(\sin^2\frac{\xi_1}{2}) = \sum_{n=0}^{N-1}\binom{N-1+n}{n}\sin^{2n}\frac{\xi_1}{2}.$$

Define ϕ by

$$\hat{\phi}(\cdot) = \prod_{j=1}^{\infty} H_0(M^{-j}\cdot).$$

Then there exists a constant C such that

$$|\hat{\phi}(\xi)| \leq C(1+|\xi_1+\xi_2|)^{-N}(1+|\xi_1|)^{-N}(1+|\xi|)^{\log_2 P_N(\frac{3}{4})},$$

and this estimate is optimal.

Proof It is easy to check that

$$|\hat{\phi}(\xi)| \leq C_1(1+|\xi_1+\xi_2|)^{-N}(1+|\xi_1|)^{-N}\prod_{j=1}^{\infty}|A(M^{-j}\xi)|. \tag{2.2}$$

Take a partition of $[-\pi, \pi]^2$ as

$$[-\pi, \pi]^2 = D_1 \cup D_2 \cup D_3 \cup D_4,$$

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where

$$\begin{aligned}
D_1 =&\{\,\xi : |\xi_1| \le \frac{2\pi}{3}, |\xi_2| \le \pi\,\}\backslash \\
&\{\xi : \xi_1 + \frac{2\pi}{3} \le \xi_2 \le \frac{2\pi}{3} - \xi_1, \xi_1 - \pi \le \xi_2 \le \xi_1 + \pi\,\}, \\
D_2 =&\{\,\xi : |\xi_1| \le \frac{2\pi}{3}, \xi_1 + \frac{2\pi}{3} \le \xi_2 \le \frac{2\pi}{3} - \xi_1, \xi_1 - \pi \le \xi_2 \le \xi_1 + \pi\,\}, \\
D_3 =&[-\pi, \pi]^2 \setminus (D_1 \cup D_2 \cup D_4), \\
D_4 =&\{\,\xi : -\pi \le \xi_1 \le -\frac{2\pi}{3}, -\frac{4\pi}{3} \le \xi_1 + \xi_2 \le -\frac{2\pi}{3}\,\} \cup \\
&\{\,\xi : \frac{2\pi}{3} \le \xi_1 \le \pi, \frac{2\pi}{3} \le \xi_1 + \xi_2 \le \frac{4\pi}{3}\,\}.
\end{aligned}$$

Since $|\xi_1| \le \frac{2\pi}{3}$ for $\xi \in D_1$,

$$|A(\xi)| = |P_N(\sin^2 \frac{\xi_1}{2})|^{\frac{1}{2}} \le (P_N(\frac{3}{4}))^{\frac{1}{2}} \tag{2.3}$$

for $\xi \in D_1$.

It is easy to check that $MD_2 \subset \{\,\xi : |\xi_1| \le \frac{2\pi}{3}\,\}$. Hence, $|A(M\xi)| \le (P_N(\frac{3}{4}))^{\frac{1}{2}}$, and consequently,

$$|A(\xi)A(M\xi)| \le (P_N(\frac{3}{4})) \tag{2.4}$$

for $\xi \in D_2$.

It is obvious that $MD_3 \subset \{\,\xi : |\xi_1| \le \frac{2\pi}{3}\,\} \cup \{\,\xi : \frac{4\pi}{3} \le |\xi_1| \le 2\pi\,\}$ and $M^2D_3 = 2D_3$. Hence,

$$\begin{aligned}
|A(\xi)A(M\xi)A(M^2\xi)| &\le |A(\xi)A(2\xi)|(P_N(\frac{3}{4}))^{\frac{1}{2}} \\
&= (P_N(\sin^2 \frac{\xi_1}{2}))^{\frac{1}{2}}(P_N(4(1-\sin^2 \frac{\xi_1}{2})\sin^2 \frac{\xi_1}{2}))^{\frac{1}{2}}(P_N(\frac{3}{4}))^{\frac{1}{2}} \\
&\le (P_N(\frac{3}{4}))^{\frac{3}{2}}
\end{aligned} \tag{2.5}$$

for $\xi \in D_3$, where [10, Lemma 7.1.8] is used in the last inequality.

By [10, Lemma 7.1.8], we obtain that

$$\begin{aligned}
|A(\xi)A(M\xi)A(M^2\xi)A(M^3\xi)| &\le |A(\xi)A(2\xi)||A(M\xi)A(2M\xi)| \\
&\le P_N(\frac{3}{4})|A(M\xi)A(2M\xi)|
\end{aligned}$$

for $\xi \in D_4$.

It is easy to check that $MD_4 \subset \{\,\xi : \frac{2\pi}{3} \le |\xi_1| \le \frac{4\pi}{3}\,\}$. Hence, $\frac{3}{4} \le \sin^2 \frac{\xi_1}{2} \le 1$ for $\xi \in D_4$, and by [10, Lemma 7.1.8], $|A(M\xi)A(2M\xi)| \le P_N(\frac{3}{4})$ for $\xi \in D_4$. Hence,

$$|A(\xi)A(M\xi)A(M^2\xi)A(M^3\xi)| \le (P_N(\frac{3}{4}))^2 \tag{2.6}$$

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for $\xi \in D_4$.

By Theorem 1.1, it follows from (2.2)-(2.6) that

$$|\hat{\phi}(\xi)| \leq C(1+|\xi_1+\xi_2|)^{-N}(1+|\xi_1|)^{-N}(1+|\xi|)^{\log_2 P_N(\frac{3}{4})}. \tag{2.7}$$

Taking $\xi^{(0)} = (-\frac{2\pi}{3}, 0)^T$, $\xi^{(1)} = (-\frac{2\pi}{3}, -\frac{2\pi}{3})^T$, $\xi^{(2)} = (\frac{2\pi}{3}, 0)^T$, and $\xi^{(3)} = (\frac{2\pi}{3}, \frac{2\pi}{3})^T$, we obtain a cycle related to M

$$\{\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}\}.$$

It easy to check that $\log_2 |A(\xi^{(m)})| = \frac{1}{2}\log_2 P_N(\frac{3}{4})$ for $m = 0, 1, 2, 3$, and consequently, by Theorem 1.2,

$$|\hat{\phi}(2^{kn_0} M\xi^{(0)})| \geq C'(1+|2^{kn_0}\xi^{(0)}|)^{-2N+\log_2 P_N(\frac{3}{4})}$$

for $k \in \mathrm{N}$. Compared with (2.7), the optimality of the estimate is showed. The proof is completed.

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关于一类二维非可分细分分布的一个注记

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摘 要: 设 $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. 本文通过引入与 M 相关的圈的概念，给出了 M- 细分分布 Fourier 变换模的一个最优的上界估计.

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