Journal of Mathematical Research & Exposition
Vol.23, No.1, 98–102, February, 2003

Periodic Solutions of Liénard Equations with Infinite Delay *

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Abstract: The problem of periodic solutions for Liénard equations with infinite delay $\ddot{x} + \frac{\partial^2 F(x)}{\partial x^2}\dot{x} + g(t, x_t) = p(t)$ is discussed by using Mawhin's coincidence degree theory. Some new results on the existence of periodic solutions are derived.

Key words: Liénard equations; infinite delay; periodic solutions.

Classification: AMS(2000) 34C25/CLC number: O175.12

Document code: A **Article ID**: 1000-341X(2003)01-0098-05

1. Introduction

The problem of periodic solutions of nonlinear Liénard equations has been taken into account greatly, which has wide background in applications (such as forced oscillations in electronic engineering, mechanical oscillation, and so on). In this paper, we discuss the existence of periodic solutions of liénard equations with infinite delay of the following form

$$\ddot{x} + \frac{\partial^2 F(x)}{\partial x^2}\dot{x} + g(t, x_t) = p(t), \tag{1}$$

where $F \in C^2(R^n, R)$, $g \in C(R \times C_h, R^n)$, $x_t \in C_h$, $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$, and g is ω-periodic with respect to the first argument, $p \in C(R, R^n)$, $p(t+\omega) = p(t)$, $\int_0^\omega p(t)\mathrm{d}t = 0$, $\omega > 0$. For details of the space C_h, we refer the readers to [1].

In recent years, with the introduction of functional analysis, some excellent results have been obtained for ordinary differential systems of the Liénard type $\ddot{x} + \frac{\partial^2 F(x)}{\partial x^2}\dot{x} + g(x) = p(t)$ on the existence of periodic solutions [2,3]. But for Eq.(1), the results on this subject have been read rarely. The purpose of this paper is to use Mawhin's coincidence degree theory to discuss the problem of periodic solutions of Eq.(1). It is convenient to solve the problem of periodic solutions for a kind of Liénard equations with exponential functions by using our results which are acquired in this paper, it is also valid to the Liénard equations with

*Received date: 2000-05-22
Foundation item: Supported by the NNSF of China (10071097)
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quasibounded nonlinearities. So our results extend the former throrems in [4].

2. Lemma and notation

Let X and Z be real normed spaces and $L : \mathrm{dom}L \subset X \to Z$ a linear Fredholm operator with index zero. Let $N : X \to Z$ be a nonlinear continuous mapping. Then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}P = \ker Q$. Moreover, the mapping $L : \mathrm{Dom}L \cap \ker P \to \mathrm{Im}L$ is invertible. Denote its inverse by $K : \mathrm{Im}L \to \mathrm{Dom}L \cap \ker P$. Let Ω be an open bounded subset of X such that $\mathrm{Dom}L \cap \Omega \neq \phi$, The mapping N is said to be L-compact on $\bar{\Omega}$ if the mappings $QN : \bar{\Omega} \to Z$ and $K(I-Q)N : \bar{\Omega} \to X$ are compact, i.e., continuous and $QN(\bar{\Omega}), K(I-Q)N(\bar{\Omega})$ are relatively compact. Let $J : \mathrm{Im}Q \to \ker L$ be an isomorphism and $M = P + JQN + K(I-Q)N$. Mawhin defined the coincidence degree in [5]: $d[(L,N),\Omega)] = \deg(I-M,\Omega,0)$. On the basis of Mawhin's theorems[5], we obtain the following lemma.

Lemma 1[6] *Suppose that the following assumptions hold:*

(1) K is continuous and N, Φ are L-compact;

(2) There exist a linear functional $\gamma : Z \to R$ with $\mathrm{Im}L \subset \ker\gamma$ and constants $\alpha_1, \alpha_2 \geq 0, \beta_1, \beta_2 \geq 0$ such that

$$|Nx| \leq \gamma(Nx) + \alpha_1|x| + \beta_1, \quad x \in X,$$

$$|\Phi x| \leq \gamma(\Phi x) + \alpha_2|x| + \beta_2, \quad x \in X,$$

(3) Every possible solution x of the equation $\lambda QNx + (1 - \lambda Q\Phi x = 0, \lambda \in (0,1)$ satisfies the relation $|Px| < \mu|(I-P)x| + r, \mu \geq 0, r > 0$;

(4) $d[(L,\Phi), B_X(s)] \neq 0, s \geq r$, where $B_X(s)$ is an open ball of center 0 and radius s in X. Then there exists an $\alpha_0 > 0$ such that the equation $Lx = Nx$ has at least one solution provided $\alpha_1 + \alpha_2 \leq \alpha_0$.

For any vector $x = (x_1, x_2, \cdots, x_n)^T$ in R^n, we define its norm $|x| = \sum_{i=1}^n |x_i|$, and for any $n \times n$ matrix $A = (a_{ij})$, define its norm $\|A\| = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$. In the following parts of this paper, we always choose $X = \{x \in C^1(R,R^n) | x(t) \equiv x(t+\omega)\}$, $Z = \{x \in C(R,R^n) | x(t) \equiv x(t+\omega)\}$. For $x \in X$, we define the norm $\|x\| = \max(\sup_{0\leq t\leq\omega} |x(t)|, \sup_{0\leq t\leq\omega} |\dot{x}(t)|)$, and for $z \in Z, |z|_1 = \frac{1}{\omega}\int_0^\omega |z(t)|dt$. It is obvious that X, Z are normed spaces. Let C denote all of continuous function $\varphi : (-\infty, 0] \to R^n, h \in C((-\infty,0],[0,\infty))$ and $0 < l = \int_{-\infty}^0 h(s)ds < \infty$, for $\varphi \in C([a,b],R^n)$,we define $|\varphi|^{[a,b]} = \sup_{a\leq t\leq b} |\varphi(t)|$. Define $C_h = \{\varphi \in C | \int_{-\infty}^0 h(s)|\varphi|^{[s,0]}ds < \infty\}$. For $\varphi \in C_h$, we define its norm $|\varphi|_h = \int_{-\infty}^0 h(s)|\varphi|^{[s,0]}ds$, and we denote the space $(C_h, |\cdot|_h)$ by C_h, then C_h is a Banach space[1].

3. Main results

Theorem 1 *Suppose that the following conditions satisfy:*

(1) There exists a constant $b > 0$ such that $\|\frac{\partial^2 F(x)}{\partial x^2}\| \leq b$ for any $x \in R^n$;

(2) The functional g maps bounded sets into bounded sets. There exist $\alpha \in R^n$, $\varepsilon \geq 0$

and continuous function $\beta : (-\infty, \infty) \to (0, \infty)$, such that

$$|g(t,\varphi)| \le \langle \alpha, g(t,\varphi)\rangle + \varepsilon|\varphi|_h + \beta(t), \quad \varphi \in C_h, t \in R,$$

where $\langle \cdot,\cdot\rangle$ denotes the inner product in R^n;

(3) There is $r > 0$ such that for any $x \in domL$ with $\inf_{t\in R}|x(t)| \ge r$, it implies that $\int_0^\omega g(t,x_t)\mathrm{d}t \ne 0$;

(4) $\deg(g_0, B_{R^n}(s), 0) \ne 0$, where $g_0 : R^n \to R^n, g_0(a) = \int_0^\omega g(s,a)\mathrm{d}s$, for every $s \ge r$. Then there is an $\alpha_0 > 0$ such that Eq.(1) has at least one ω-periodic solution provided $(b + 2\varepsilon l) < \alpha_0$.

Proof Define $Lx = -\ddot{x}$, $Nx = \frac{\partial^2 F(x)}{\partial x^2}\dot{x} + g(t,x_t) - p(t)$. It is easy to see that L is a Fredholm operator with index 0, and for any subset Ω of X, N is L-compact On $\bar{\Omega}$, and $\ker L = R^n$. Define the projections $Px = \frac{1}{\omega}\int_0^\omega x(t)\mathrm{d}t$ for any $x \in X$, $Qz = \frac{1}{\omega}\int_0^\omega z(t)\mathrm{d}t$ for any $z \in Z$. If we define the linear functional $\gamma : Z \to R$ by $\gamma(z) = \langle \alpha, \frac{1}{\omega}\int_0^\omega z(t)\mathrm{d}t\rangle$, then $\mathrm{Im}L \subset \ker\gamma$.
Since

$$\begin{aligned}|Nx|_1 =& \frac{1}{\omega}\int_0^\omega |Nx(t)|\mathrm{d}t = \frac{1}{\omega}\int_0^\omega \left|\frac{\partial^2 F(x)}{\partial x^2}\dot{x} + g(t,x_t) - p(t)\right|\mathrm{d}t\\ \le& \frac{1}{\omega}\int_0^\omega \left|\frac{\partial^2 F(x)}{\partial x^2}\dot{x}\right|\mathrm{d}t + \frac{1}{\omega}\int_0^\omega |g(t,x_t)|\mathrm{d}t + \frac{1}{\omega}\int_0^\omega |p(t)|\mathrm{d}t\\ \le& b\|x\| + \frac{1}{\omega}\int_0^\omega |g(t,x_t)|\mathrm{d}t + \frac{1}{\omega}\int_0^\omega |p(t)|\mathrm{d}t,\end{aligned}$$

and

$$\gamma(Nx) = \langle \alpha, \int_0^\omega Nx(t)\mathrm{d}t\rangle = \langle \alpha, \int_0^\omega g(t,x_t)\mathrm{d}t\rangle,$$

from condition(2), we can see

$$|Nx|_1 \le \gamma(Nx) + \alpha_1\|x\| + \beta_1,$$

where $\alpha_1 = b + \varepsilon l, \beta_1 = \frac{1}{\omega}\int_0^\omega |p(t)|\mathrm{d}t + \frac{1}{\omega}\int_0^\omega \beta(t)\mathrm{d}t$.

Define $\Phi x = \frac{1}{\omega}\int_0^\omega g(t,x_t)\mathrm{d}t = QNx(t)$, we know that

$$\begin{aligned}|\Phi x|_1 =& \frac{1}{\omega}\int_0^\omega g(t,x_t)\mathrm{d}t \le \gamma(\Phi x) + \varepsilon l\|x\| + \frac{1}{\omega}\int_0^\omega \beta(t)\mathrm{d}t\\ =& \gamma(\Phi x) + \alpha_2\|x\| + \beta_2,\end{aligned}$$

where $\alpha_2 = \varepsilon l, \beta_2 = \frac{1}{\omega}\int_0^\omega \beta(t)\mathrm{d}t$.

For every possible solution of the following equation

$$(1-\lambda)QNx + \lambda Q\Phi x = QNx = 0,$$

i.e., $\int_0^\omega g(t,x_t)\mathrm{d}t = 0$, from condition(3), there is a number $\sigma \in [0,\omega)$, such that $|x(\sigma)| < r$. Since Px is constant value function, $\|Px\| = |Px(\sigma)|$,$|x(\sigma)| = |Px(\sigma) + (I-P)x(\sigma)| \ge |Px(\sigma)| - |(I-P)x(\sigma)|$, we obtain

$$\|Px\| \le |x(\sigma)| + |(I-P)x(\sigma)| \le r + \|(I-P)x\|.$$

Now we consider the equation $Lx = \Phi x$. From $\mathrm{Im}L = \ker Q, \Phi = QN$, we know that $Lx = \Phi x$ is equivalent to the following two equations

$$QNx = 0, \quad Lx = (I - Q)QNx = 0,$$

i.e., $QNx = 0$ and $x \in \ker L$, from condition(3), we can see $Lx \neq \Phi x$, for $x \in \ker L \cap \partial B_X(s)$ and $s \geq r$. Since $\mathrm{Im}Q = \ker L = R^n$, we define the isomorphism $J = I : \mathrm{Im}Q \to \ker L$, from proposition II.12 in [7] and condition(4), we have

$$|d[(L, \Phi), B_X(s)]| = |\deg(JQN, B_X(s) \cap \ker L, 0)| = |\deg(g_0, B_{R^n}(s), 0)| \neq 0,$$

where $s \geq r$. It is obvious that lemma 1(1) is satisfied. From the above statements, all of the conditions of lemma 1 are hold. Thus there exists a constant $\alpha_0 > 0$ such that Eq.(1) has at least one ω-periodic solution provided $(b + \varepsilon l) \leq \alpha_0$. Theorem 1 is proved.

Corollary *If we replace condition(4) in theorem 1 with the following:*

There exists a function $V(x) \in C^1(R^n, R)$, $\lim_{|x| \to \infty} V(x) = \infty$, *such that* $\mathrm{grad}V(x) \neq 0$, *and* $\langle \mathrm{grad}V(x), \int_0^\omega g(t, x_t)\mathrm{d}t \rangle > 0$ *for* $|x| \geq r$.

Then the conclusion of Theorem 1 remains valid.

Proof It is only necessary to prove $\deg(g_0, B_{R^n}(s), 0) \neq 0$, for $s \geq r$. In fact, from condition(3) of theorem 1, we can see $QNx = \int_0^\omega g(t, x_t)\mathrm{d}t \neq 0$ for $x \in R^n$ and $|x| \geq r$. We define $H(\tau, x) = (1 - \tau)QNx + \tau \mathrm{grad}V(x), \tau \in [0, 1]$, then $\langle \mathrm{grad}V(x), H(\tau, x) \rangle > 0$ for $x \in B_{R^n}(s)$. From the homotopy invariance of degree and Theorem 6.3 in [8], we have $\deg(g_0, B_{R^n}(s), 0) = \deg(gradV, B_{R^n}(s), 0) = 1$.

Example Consider the equation

$$\begin{aligned} \ddot{x} + \frac{\varepsilon \dot{x}}{\sqrt{(1 + x^2)^3}} + \varepsilon \int_{-\infty}^0 e^\theta x_t(\theta)\mathrm{d}\theta \exp\left(\int_{-\infty}^0 e^\theta x_t(\theta)\mathrm{d}\theta\right) - \frac{\varepsilon(\sin t - \cos t)}{2} \exp\left(\frac{\sin t - \cos t}{2}\right) \\ = \frac{\varepsilon \cos t}{\sqrt{(1 + \sin t^2)^3}} - \sin t, \end{aligned} \tag{2}$$

we choose $\alpha = 1, b = \varepsilon > 0$, $F(x) = \varepsilon\sqrt{1 + x^2}$, $p(t) = \frac{\varepsilon \cos t}{\sqrt{(1 + \sin t^2)^3}} - \sin t$, $g(t, \varphi) = \varepsilon \int_{-\infty}^0 e^\theta \varphi(\theta)\mathrm{d}\theta e^{\int_{-\infty}^0 e^\theta \varphi(\theta)\mathrm{d}\theta} - \frac{\varepsilon(\sin t - \cos t)}{2} e^{\frac{\sin t - \cos t}{2}}$, $\beta(t) = \varepsilon|\sin t - \cos t|e^{\frac{\sin t - \cos t}{2}}$, we can easily see all of conditions of theorem 1 are satisfied, so Eq.(2) has at least one 2π-periodic solution. In fact, $x(t) = \sin t$ is an obvious 2π-periodic solution of Eq.(2).

Remark In the above example, the functional $g(t, \varphi)$ doesn't hold the following condition: there exist constants $\varepsilon, \beta \geq 0$ such that $|g(t, \varphi)| \leq \varepsilon|\varphi|_h + \beta$. So the results in [4] are invalid to the equations with exponential function.

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无穷时滞 Liénard 方程的周期解

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摘 要: 讨论具有无穷时滞 Liénard 型方程 $\ddot{x}+\frac{\partial^2 F(x)}{\partial x^2}\dot{x}+g(t,x_t)=p(t)$ 的周期解问题, 利用重合度理论得到了周期解存在的充分条件.