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A Note on AP-Injective Rings *

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Abstract: The purpose of this paper is to study the following two questions on AP-injective rings: (1) R is a regular ring if and only if R is a left PP-ring and R is left AP-injective; (2) Let R be a right AP-injective ring. Then R is self-injective if and only if R is weakly injective. Hence we get some new results of P-injective rings.

Key words: AP-injective rings; weakly injective rings; PP-rings.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. For any nonempty subset X of a ring R, $r(X)$ and $l(X)$ are reserved for the right annihilator of X and the left annihilator of X respectively. From Page and Zhou (cf.[1]), a ring R is called right AP-injective if for any $(0\neq)a\in R$ there exists a left ideal X_a of R such that $lr(a)=Ra\oplus X_a$. Similarly, we have the conceptions of left AP-injective rings. It is well known that R is right P-injective if and only if $lr(a)=Ra$, for any $a\in R$ (cf. [2]). Hence P-injective is AP-injective, but AP-injective rings need not be GP-injective rings ([1], Example 1.5),also not be P-injective rings. Following [3], Ring R is called a left (resp right) PP-ring if for any $a\in R$, $Ra(aR)$ is projective; R is said to be a left (resp right) GPP-ring if for any $a\in R$, there exists a positive integer m such that $Ra^m(a^mR)$ is a projective R-module. In [4], Xue Wei Ming prove that R is regular if and only if R is left P-injective, left PP-ring. Recently, Professor Chen and Ding have prove that R is a regular ring if and only if R is a PP-ring and ${}_RR$ is GP-injective (cf.[5]). In the meantime, they obtained a ring which is left P-injective, right PP-ring is not regular. This paper we will prove a more detailed result that R is a regular ring if and only if R is a left AP-injective, left PP-ring. Whence, we have the conclusion that if R is a right PP-ring,

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then R is a left AP-injective ring if and only if R is a left P-injective ring if and only if R is regular. In [6], a module M is said to satisfy (C_2) if for any two submodules X and Y of M with X is a direct summand of M and $Y \cong X$, we have Y is a direct summand of M. Here we show that for a left AP-injective ring, if $Re \cap Rf = 0$ with $e^2 = e, f^2 = f$ are in R, then there exists $g^2 = g \in R$ such that $Re \oplus Rf = Rg$ by the C_2 condition.

2. PP-rings

Lemma 2.1 *Let R be a left PP-ring. If R is left AP-injective , then R is regular.*

Proof For any $a \in R$, we have a short exact sequence

$$0 \to l(a) \to R \to Ra \to 0.$$

Where $\varphi : R \to Ra; r \mapsto ra$. Since R is left PP-ring, so Ra is projective. Then there exists $L \leq_R R$ such that ${}_RR = l(a) \oplus L$, so $l(a) = Re$ with $e^2 = e \in R$. We have $rl(a) = r(Re) = (1-e)R$. Denotes $f = 1 - e$, then $rl(a) = fR, f^2 = f \in R$. Inasmuch as $a \in rl(a)$, so $a = fa$. But R is left AP-injective ring, there exists $X_a \leq R_R$ such that

$$rl(a) = aR \oplus X_a.$$

So $f = ar + x$ with $r \in R, x \in X_a$, then

$$xa = fa - ara = a - ara = a(1 - ra) \in aR \cap X_a = 0.$$

Hence $a = ara$, R is regular.

Corollary 2.2 *Let R be a left PP-ring. If R is a left AP-injetive ring, then R is a right PP-ring.*

Proof By Lemma 2.1, R is regular. For any $a \in R$, we have $e^2 = e \in R$ such that $aR = eR$. Hence $aR \oplus (1-e)R = R_R$, and aR is projective. R is right PP-ring.

Theorem 2.3 *R is a regular ring if and only if R is a left AP-injective, left PP-ring.*

Proof By Lemma 2.1, R is regular. Converse,since R is regular, $Ra = Rf$ with $f^2 = f \in R$ for any $a \in R$, so R is left PP-ring. Likewise, $aR = eR$ with $e^2 = e \in R$,

$$aR = eR = rl(eR) = rl(aR) = rl(a),$$

R is left AP-injective. □

Remark As the proof in Lemma 2.1, we have the same result for a right AP-injective, right PP-ring. Whence, for a left PP-ring, R is left AP-injective if and only if R is left P-injective if and only if R is regular.

Ring R is π-regular ring if for any $a \in R$, there exist $b \in R$ and positive integer m such that $a^m = a^m b a^m$. Clearly, regular rings are π-regular rings, but π-regular rings need not be regular rings as the following example.

Example 2.4 Let

$$R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad a,b,c \in Z_2.$$

R is π-regular, but since $J(R) = R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix} \neq 0$, so R is not regular.

In [7], we have the conclusion that for a right AP-injective ring, if R is a right GPP-ring, then R is a π-regular ring and R is a left GPP-ring. Of course, we have the similar conclusion, i.e. if R is a left AP-injective, left GPP-ring then R is a π-regular ring and R is a right GPP-ring as the following Proposition

Proposition 2.5 *Let R be a left AP-injective ring. If R is a left GPP-ring, then R is π-regular.*

Proof For any $a \in R$, there exists a positive integer m such that Ra^m is projective. So we have a short exact sequence

$$0 \to l(a^m) \to R \to Ra^m \to 0.$$

Where $\varphi : R \to Ra^m; r \mapsto ra^m$. Since R is left GPP-ring, so Ra^m is projective. Then there exists $L \leq_R R$ such that ${}_RR = l(a^m) \oplus L$, so $l(a^m) = Re$ with $e^2 = e \in R$. We have $rl(a^m) = r(Re) = (1-e)R$, and $ea^m = 0$. Inasmuch as R is left AP-injective ring, there exists $X_{a^m} \leq R_R$ such that

$$rl(a^m) = a^m R \oplus X_{a^m}.$$

So $1 - e = a^m r + x$ with $r \in R, x \in X_{a^m}$, then

$$xa^m = (1-e)a^m - a^m r a^m = a^m(1 - ra^m) \in a^m R \cap X_{a^m} = 0.$$

Hence $a^m = a^m r a^m$, R is π-regular.

Corollary 2.6 *Let R be a left AP-injective ring. If R is a left GPP-ring, then R is a right GPP-ring.*

R is called a Baer ring if the right annihilator of every nonempty subset of R is generated by an idempotent (cf.[8]).

Proposition 2.7 *Let R be a left AP-injective ring. If R is Baer ring, then R is regular ring.*

Proof For any $a \in R$, since $\emptyset \neq l(a) \subseteq R$, and R is Baer ring, then $rl(a) = eR$ with $e^2 = e \in R$. So $a = ea$. But R is left AP-injective, then

$$aR \oplus X_a = rl(a) = eR, \qquad X_a \leq R_R.$$

Hence there exist $b \in R, x \in X_a$ such that

$$e = ab + x, a = ea = aba + xa, a(1 - ba) = xa \in X_a \cap aR = 0.$$

Whence $a = aba$, R is regular.

Corollary 2.8 *Let R be a left AP-injective ring. If R is Baer ring, then R is left PP-ring and right PP-ring, and R is a right AP-injective ring.*

3. Weakly injective rings

Let $E(M)$ is an injective hull of M_R. M is called weakly injective (cf.[9]) if for any finite generated submodule $N_R \subseteq E(M)$, there exists $X_R \cong M$, and $N_R \subseteq X_R \subseteq E(M)$. Clearly, injective rings are weakly injective rings, but the converse need not true (cf.[9]). While, weakly injective rings are injective rings for AP-injective rings as follows

Lemma 3.1 *Let R be a right AP-injective ring. If $R_R \leq_e X \cong R_R$, then $X = R$, where X is right R-module.*

Proof Denotes φ as the isomorphism of R_R to X with $\varphi(1) = b \in X$, then $Im\varphi = X = bR$. And $1 \in R \leq_e X$. So there exists $u \in R$, and $1 = bu$. Hence $R_R = 1R = buR$, and $r(u) = 0$. In fact, if $ur = 0$, then $bur = 0, r = 1r = bur = 0$. We have $r(u) = 0$. i.e. $R = lr(u)$. And R is right AP-injective ring, so

$$lr(u) = Ru \oplus X_u, X_u \leq_R R.$$

Hence $R = Ru \oplus X_u$, so there exist $v \in R, x \in X_u$ such that

$$1 = vu + x, \quad u = uvu + ux, \quad ux = (1 - uv)u \in X_u \cap Ru = 0,$$

$u = uvu$. Let $e = uv \in R$, so $e^2 = e$, and $uR = eR$. Then we have $R = buR = beR$, but $X = bR = b(eR + (1 - e)R) = beR + b(1 - e)R$. If $x \in beR \cap b(1 - e)R$, then there exist $r_1, r_2 \in R$ such that $x = ber_1 = b(1 - e)r_2$, so $\varphi^{-1}(x) = er_1 = (1 - e)r_2$, and since $er_1 = e(1 - e)r_2 = 0$, i.e.$x = 0$, so $X = bR = beR \oplus b(1 - e)R$. And $R_R \leq_e X$, $R = beR$, whence $b(1 - e)R = 0, X = beR = R$. □

Theorem 3.2 *Ring R is right self-injective if and only if R is right AP-injective and R_R is weakly injective.*

Proof We need only to prove $E(R_R) \subseteq R$. For any $a \in E(R_R)$, since $R + aR \subseteq E(R_R)$ and R_R is weakly injective, then there exists $X \leq E(R_R)$ such that $R + aR \leq X$ and $X \cong R$. Since R_R is right AP-injective, by Lemma 3.1, $X = R_R$. Hence $R = E(R_R)$. R is right self-injective. □

Remark We have the conclusion that if ${}_RR \leq_e X \cong_R R$, then ${}_RX = R$ for a left AP-injective as the proof in Lemma 3.1, so R is a left injective ring if and only if R is a left AP-injective, weakly injective ring.

Proposition 3.3 *Let R be a left AP-injective ring, then*

(1) If $e^2 = e \in R$, $\varphi : Re \to Ra$ is a left R-isomorphism, then there exists $f^2 = f \in R$ such that $Ra = Rf$.

(2) If $e^2 = e \in R$, $f^2 = f \in R$ and $Re \cap Rf = 0$, then there exists $g^2 = g \in R$ such that $Re \oplus Rf = Rg$.

Proof (1). Let $\varphi(e) = b$, then $Ra = Im\varphi = Rb$. Since

$$\varphi : Re \to R\varphi(e); re \mapsto r\varphi(e)$$

is R-isomorphic, so $l(e) = l(\varphi(e)) = l(b)$. Hence $b \in rl(b) = rl(e) = rl(eR) = r(R(1-e)) = eR$, $b = eb$. But R is left AP-injective, so

$$rl(b) = bR \oplus X_b, \quad X_b \leq R_R.$$

Then $e = bd + x$ with $d \in R, x \in X_b$. Hence we have $b = eb = bdb + xb$, and

$$xb = b - bdb = b(1 - db) \in bR \cap X_b = 0.$$

Whence $b = bdb$. Let $f = db$, then $Ra = Rb = Rf$.

(2). By the supposition, Re is a direct summand of R, there exists $L_1 \leq_R R$ such that ${}_RR = Re \oplus L_1$. Whence

$$Re \oplus Rf = (Re \oplus Rf) \cap R = (Re \oplus Rf) \cap (Re \oplus L_1) = Re \oplus ((Re \oplus Rf) \cap L_1).$$

Then $Rf \cong (Re \oplus Rf)/Re \cong (Re \oplus Rf) \cap L_1$. By (1), $(Re \oplus Rf) \cap L_1$ is generated by an idempotent of ${}_RR$. So there exists $L_2 \leq_R R$ such that

$$((Re \oplus Rf) \cap L_1) \oplus L_2 =_R R,$$

$$L_1 = L_1 \cap R = L_1 \cap (((Re \oplus Rf) \cap L_1) \oplus L_2) = ((Re \oplus Rf) \cap L_1) \oplus (L_1 \cap L_2).$$

Hence

$${}_RR = Re \oplus L_1 = Re \oplus ((Re \oplus Rf) \cap L_1) \oplus (L_1 \cap L_2) = Re \oplus Rf \oplus (L_1 \cap L_2),$$

$Re \oplus Rf$ is a direct summand of R, so there exists $g^2 = g \in R$ such that

$$Re \oplus Rf = Rg.$$

Corollary 3.4 *If R is a left AP-injective ring, then R satisfies the condition C_2.*

Proposition 3.5 *If R is a left AP-injective ring, for any $a \in R$, then the following conditions are equivalent:*

(1) Ra is projective as a left R-module.

(2) Ra is a direct summand of ${}_RR$.

(3) Ra is P-injective as a left R-module.

Proof (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) , obviously. (3) $\Rightarrow$ (2). Since we have left homomorphism

$$f : Ra \to Ra; ra \mapsto ra.$$

And Ra is P-injective, then there exists $ba \in Ra$ such that $a = f(a) = aba$. Let $e = ba$, then $e^2 = e$ and $Ra = Re$. Hence Ra is a direct summand of ${}_RR$.

(1)⇒ (2). Consider the following short exact sequence

$$0 \to l(a) \to R \to Ra \to 0.$$

Since Ra is projective, there exists $T \cong Ra$, and ${}_RR = l(a) \oplus T$. By Corollary 3.4, R satisfies C_2., Ra is a direct summand of ${}_RR$. Then we have complete it. □

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References:

[1] STANLEY S P, ZHOU Yi-qiang. *Generalizations of principally injective rings* [J]. J.Algebra, 1998, **206**: 706–721.

[2] NICHOLSON W K, YOUSIF M F. *Principally injective rings* [J]. J. Algebra, 1995, **174**: 77–93.

[3] HIRANO Y. *On generalized PP-rings* [J]. Math. J. Okayama Univ., 1983, **25**: 7–11.

[4] XUE Wei-min. *On PP-rings* [J]. Kobe. J. Math., 1990, **7**: 77–80.

[5] CHEN Jian-long, DING Nan-qing. *On regularity of rings* [J]. Algebra Colloquium, 2001, **8**(3): 267–274.

[6] MOHAMMED S H, MULLER B J. *Continuous and Discrete Modules* [M]. London: Cambridge Univ. Press, 1990.

[7] XIAO Guangshi. *A note on AP-injective rings* [J]. Journal of Anhui Normal University(Natural Science), 2001, **3**: 210–213.

[8] ZHANG Ju-le. *Fully idempotent rings whose every maximal left ideal is an ideal* [J]. Chinese Sci.Bull., 1992, **37**(13): 1065–1068.

[9] JAIN S K, LOPEZ-PERMOUTH S R. *Rings whose cyclics are essentially embeddable in projective modules* [J]. J. Algebra, 1990, **128**: 257–269.

关于 *AP*- 内射环的一个注记

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摘 要: 本文的主要目的是讨论 AP- 内射环中的两个问题: (1) 环 R 是正则的当且仅当 R 是左 AP- 内射的左 PP- 环; (2) 如果 R 是左 AP- 内射环, 那么 R 是内射环当且仅当 R 是弱内射环. 因此我们推广了内射环的一些结果, 与此同时我们还取得了一些新的结果.

关键词: AP- 内射环; 弱内射环; PP- 环.