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The f-width in Locally Convex Spaces *

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Abstract: In this paper we extend the width problems in normed space to locally convex space and some results are given.

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1. Introduction

Let X be a locally convex space, f a function on X, A a subset of X and n an integer with $0 \leq n < \infty$. The number

$$d_n(f,A) = \inf_{\dim G=n} \sup_{x\in A} \inf_{g\in G} f(x-g),$$

where the inf is taken over all n-dimensional linear subspaces G of X, is called f-n- width of the subset A .

A n-dimensional subspace G of X is called a best n-dimensional secant of A(with respect to X), if

$$d_n(f,A) = \sup_{x\in A} \inf_{g\in G} f(x-g).$$

Let f be a continuous convex function on X. We assume there exists a continuous bijection $\psi : R_+ \longrightarrow R_+$ $(R_+ = [0,+\infty))$, such that

(F$_1$) $f(\lambda x) = \psi(\lambda) f(x)$ $(\forall \lambda \geq 0, x \in X)$.

If f is a real function,for any $r > 0, x \in X$, let

$$P_r(x) = \inf\{t > 0 : x \in tS_r\},$$

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where $S_r = \{x \in X : f(x) \le r\}$. Obviously, if f is a continuous convex function, and $f(0) = 0$, for any $r > 0$, P_r is the *Minkowski* functional decided by the convex absorbing set S_r, and P_r is continuous positive homogeneitive and sub-additive.

2. Some properties

Lemma 1[2] *Let X be a locally convex space, f a nongative continuous convex function satisfying the condition (F_1), $f(0) = 0$, then for any $\lambda, r > 0$, we have $P_r = \lambda P_{\psi(\lambda)r}$.*

Remark Given $\forall \alpha, r > 0$, we have $P_r = \alpha P_{\psi(\alpha)r}$, also $\forall \lambda > 0$, suppose $\alpha = \psi^{-1}(\lambda/r)$, so we have $P_r = \psi^{-1}(\lambda/r)P_\lambda$.

Lemma 2 *Let X be locally convex space, f a continuous convex function satisfying the condition (F_1), $f(0) = 0$, and there exists an $x_0 \in X$, $f(x_0) > 0$. Where the ψ is the function in (F_1), then*

1. *$\psi(0) = 0, \psi(1) = 1$;*
2. *ψ is a strictly increasing function on R_+, so the converse ψ^{-1} exists and is continuous;*
3. *ψ is a convex function;*
4. *$\lim_{\lambda \to \infty} \psi(\lambda) = \infty$;*
5. *$\psi(\lambda)\psi(1/\lambda) = \psi^{-1}(\lambda)\psi^{-1}(1/\lambda) = 1$ $(\lambda > 0)$.*

Proof 1-4 have been given by [4], so we prove 5 only.

Since f is a non-zero function,there exists an $x_0 \in X$ such that $f(x_0) \neq 0$. For any $\lambda > 0$, such that

$$f(x_0) = f(\lambda \frac{1}{\lambda} x_0) = \psi(\lambda) f(\frac{1}{\lambda} x_0) = \psi(\lambda)\psi(1/\lambda) f(x_0),$$

namely $\psi(\lambda)\psi(1/\lambda) = 1$. Let $\alpha = \psi^{-1}(\lambda), \beta = \psi^{-1}(1/\lambda)$. Then

$$\begin{aligned} 1 = \psi(\alpha)\psi(\beta) = \psi(\alpha)\psi(1/\alpha) \Longrightarrow \psi(1/\alpha) = \psi(\beta) \\ \Longrightarrow \beta = 1/\alpha \Longrightarrow \psi^{-1}(\lambda)\psi^{-1}(1/\lambda) = 1. \end{aligned}$$

Theorem 3 *Let X be a locally convex space, A a subset of X and f a nonnegative continuous and convex function, $f(0) = 0$, $0 \le n < \infty$, then*

1. *We have $d_n(f, A) = d_n(f, \bar{A})$, $\bar{A}$ is the closed hull of A;*
2. *For $\forall \alpha > 0$, $d_n(f, \alpha A) = \psi(\alpha) d_n(f, A)$;*
3. *For the circled hull $\tau(A)$ of A, we have $d_n(f, \tau(A)) = d_n(f, A)$;*
4. *For the convex hull $co(A)$ of A, we have $d_n(f, co(A)) = d_n(f, A)$;*
5. *We have $d_0(f, A) \ge d_1(f, A) \ge \cdots \ge d_n(f, A) \ge \cdots$;*
6. *If A is compact, we have $\lim_{n \to \infty} d_n(f, A) = 0$;*
7. *If $\dim(\text{span } A) = n$, we have $d_n(f, A) = d_{n+1}(f, A) = \cdots = 0$.*

Proof We ignore the proof of 1-5.

6. Let A be compact and let $\varepsilon > 0$ be arbitrary. Take an $f-\varepsilon-$net $\{x_1, \cdots x_N\}$ for A and let $G = \text{span}\{x_1, \cdots x_N\}$. then for every $x \in A$ we have

$$\inf_{g \in G} f(x-g) \leq \min_{1 \leq i \leq N} f(x - x_i) < \varepsilon,$$

hence

$$\inf_{\dim G = N} \sup_{x \in A} \inf_{g \in G} f(x - x_i) \leq \varepsilon.$$

whence, by 5, there follows 6.

7. Since

$$d_n(f, A, X) = \inf_{\dim G = n} \sup_{x \in A} \inf_{g \in G} f(x-g) \leq \sup_{x \in \text{span} A} \inf_{g \in G} f(x-g) = 0.$$

which, taking into account 5, completes the proof.

In order to get the uniformity of f and P_r on the width, we give the following lemma.

Lemma 4 *Let f be a continuous convex function, and $f(0) = 0 \leq f(x)$, for every $r > 0$, we have*

1. $f(x) \leq r \Longleftrightarrow P_r(x) \leq 1$;
2. $f(x) = r \Longleftrightarrow P_r(x) = 1$;
3. $f(x) \geq r \Longleftrightarrow P_r(x) \geq 1$.

3. An application

Let X be a locally convex space and G a subspace of X, note

$$P_{f,G}(x) = \{g \in G : f(x-g) = \inf f(x-y), \forall y \in G\},$$

in the case when this will lead to no confusion, we shall use $P_f(x)$.

Remark Suppose $0 \in P_f(x)$, for every $\alpha > 0$ and every $g \in G$, satisfying

$$f(\alpha x) = \psi(\alpha) f(x) \leq \psi(\alpha) f(x-g) = f(\alpha x - \alpha g) \leq f(\alpha x - g_1) \quad (g_1 = \alpha g \in G)$$

hence $0 \in P_f(\alpha x)$. We use the notation $x \perp G$ for $0 \in P_f(x)$.

In order to get the following results, assume

(F_2) f satisfies (F_1), and $f(-x) = f(x)$.

Lemma 5 *Let X be a locally convex space and G_1, G_2 two linear subspaces of X such that*

$$\dim G_1 < \infty, \dim G_1 < \dim G_2,$$

f is a function on X satisfying (F_2), then there exists a $y \in G_2 \backslash \{0\}$, such that $y \perp G_1$.

Proof Obviously we may assume, without loss of generality, that we have

$$\dim G_1 = n, \quad \dim G_2 = n+1.$$

Let $X_1 = \text{span}\{G_1, G_2\}$ = the linear subspace of X spanned by $G_1 \cup G_2$. As we all know that f and P_r have the same approximation properties[4], then we consider P_r in instead of f.

Since X_1 is a finite dimensional subspace of X, there exists a norm $\|\cdot\|_*$ on X_1. For every $\varepsilon > 0$, let

$$P_r(x) + \varepsilon \| x \|_* = \| x \|_\varepsilon \qquad (\varepsilon > 0).$$

then $\| * \|_\varepsilon$is a norm on X_1. Subsequently we prove the lemma is right for P_r.

By Iven Singer[2], for $\|\cdot\|_\varepsilon$, there exists a $y \in G_2\backslash\{0\}$, such that $y \perp G_1$. Then $\beta y \perp G_1$, for any $\beta > 0$.

Take $\varepsilon = \frac{1}{n}$, then there exist $y_n \in G_2\backslash\{0\}$, such that $\| y_n \|_* = 1$, and for $\|\cdot\|_{\frac{1}{n}}$, $y_n \perp G_1$. Since X_1 is finite dimensional. Choosing a convergent subsequence, we can assume $y_n \longrightarrow y$, then $\| y \|_* = 1$, and $y \neq 0$. Since G_2 is finite dimensional, G_2 is closed. So we have $y \in G_2$, for every $g \in G_1$, since

$$\begin{aligned} \| y_n \|_{\frac{1}{n}} &= P_r(y_n) + \frac{1}{n} \| y_n \|_* = P_r(y_n) + \frac{1}{n} \\ &\leq \| y_n - g \|_{\frac{1}{n}} = P_r(y_n - g) + \frac{1}{n} \| y_n - g \|_*, \end{aligned}$$

let $n \longrightarrow \infty$, we have

$$P_r(y) \leq P_r(y - g).$$

Namely $y \perp G_1$, which completes the proof of Lemma 5.

Theorem 6 *Let X be a locally convex space and X_{n+1} a $n+1$-dimensional subspace of X, f satisfies (F_2), and when $x \in X_{n+1}\backslash\{0\}$, $f(x) \neq 0$, we have*

$$d_n(f, S_{X_{n+1}}) = 1.$$

Proof Let G be an n-dimensional subspace of X. By Lemma 10, let $G_1 = G$, $G_2 = X_{n+1}$, then there exists a $y_0 \in X_{n+1}\backslash\{0\}$, such that $y_0 \perp G$, whence, for every $g \in G$, such that

$$f\left(\frac{y_0}{\psi^{-1}(f(y_0))} - g\right) = \frac{1}{f(y_0)} f(y_0 - \psi^{-1}(f(y_0))g) \geq \frac{1}{f(y_0)} f(y_0) = 1.$$

Hence

$$1 \geq \sup_{x \in S_{X_{n+1}}} \inf_{g \in G} f(x - g) \geq \inf_{g \in G} f\left(\frac{y_0}{\psi^{-1}(f(y_0))} - g\right) = 1,$$

namely

$$\sup_{x \in S_{X_{n+1}}} \inf_{g \in G} f(x - g) = 1.$$

Whence, since G was an arbitrary $n-$dimensional subspace of X, which completes the proof of Theorem 6.

4. Extension of the application

First we assume that ψ has the condition $f(\lambda x) = \psi(|\lambda|)f(x)$. If there is an $x \in X$ such that $f(x) > 0$, by Lemma 2, for any $s, t > 0$, we have $\psi(st) = \psi(s)\psi(t)$.

Lemma 7 *Let X be a locally convex space and f a function on X satisfying (F_1), $0 = f(0) < f(x)$ $(x \neq 0)$, $S_X = \{x : f(x) < 1\}$ is bounded. For any linear nonzero continuous functional φ, there exists a $r > 0$ such that*

$$\sup_{x \neq 0} \frac{\psi(r|\varphi(x)|)}{f(x)} = 1.$$

Proof Since $D = \{x : |\varphi(x)| < 1\}$ is open and S_X bounded, there exists a $M > 0$ such that $S_X \subseteq MD$. So we have $|\varphi(x)| \leq M$ when $f(x) \leq 1$. Put

$$\sup_{f(x)=1} \frac{\psi(|\varphi(x)|)}{f(x)} = \rho,$$

obviously, $0 < \rho < +\infty$. For any $x \neq 0$, there exist a $\lambda > 0$ such that $f(\lambda x) = \psi(\lambda)f(x) = 1$.(We can assume $\lambda = \psi^{-1}(1/f(x))$) Then for any $x \neq 0$, we have

$$\frac{\psi(|\varphi(x)|)}{f(x)} = \frac{\psi(\lambda^{-1}|\varphi(\lambda x)|)}{f(\lambda^{-1}(\lambda x))} = \frac{\psi(\lambda^{-1})\psi(|\varphi(\lambda x)|)}{\psi(\lambda^{-1})f(\lambda x)} \leq \rho.$$

Put

$$\sup_{x \neq 0} \frac{\psi(|\varphi(x)|)}{f(x)} = \frac{\psi(|\varphi(x)|)}{f(x)} = \rho_0 = \psi(r^{-1}) = 1/\psi(r).$$

Let $r^{-1} = \psi^{-1}(\rho_0)$ Hence

$$\sup_{x \neq 0} \frac{\psi(r|\varphi(x)|)}{f(x)} = \sup_{x \neq 0} \frac{\psi(r)\psi(|\varphi(x)|)}{f(x)} = 1.$$

In order to get more results, we define f-distance in locally convex space X. Given $x, y \in X$, the distance between x and y is defined by $\rho_f(x, y) = f(x - y)$. If $H = \{x : \varphi(x) = \alpha\}$ is a hyperplane, the distance from x to H is defined by $\rho_f(x, H) = \inf_{y \in H} f(x - y)$. Then we have

Lemma 8 *X, f, φ are same to that of lemma 7. $H = \{x : \varphi(x) = \alpha\}$is a hyperplane. Assume $\sup_{x \neq 0} \frac{\psi(|\varphi(x)|)}{f(x)} = \beta > 0$. For any $x \in X$, we have*

$$\rho_f(x, H) = \frac{1}{\beta}\psi(|\varphi(x) - \alpha|).$$

Proof For any $y \in H$. we have

$$f(x - y) \geq \frac{1}{\beta}\psi(|\varphi(x - y)|) = \frac{1}{\beta}\psi(|\varphi(x) - \alpha|),$$

whence $\rho_f(x,H) \geq \frac{1}{\beta}\psi(|\varphi(x)-\alpha|)$. On the other hand, if $0<\lambda<\beta$, there exists a $z \in X$ such that

$$\psi(|\varphi(z)|) > (\beta-\varepsilon)f(z).$$

Putting

$$y = x - \frac{\varphi(x)-\alpha}{\varphi(z)}z,$$

we obtain $\psi(|\varphi(x)-\alpha|) > (\beta-\varepsilon)f(x-y)$, hence

$$f(x-y) = \frac{\psi(|\varphi(x)-\alpha|)}{\beta-\varepsilon}.$$

Since $\varepsilon>0$ was arbitrary and $y \in H$, it follows that we have $\rho_f(x,H) \leq \frac{1}{\beta}\psi(|\varphi(x)-\alpha|)$, which, together with the opposite inequality shown above, complete the proof of lemma 8.

In locally convex space, if A is a bounded closed circle convex set, obviously, for any $\lambda>0$, λA is also the same set to A. We can obtain $\partial(\lambda A) = \lambda\partial A$. $x \in \partial A$. There exist two sequences $x_n \in A$, $y_n \notin A$, such that $x_n \longrightarrow x$, $y_n \longrightarrow y$, obviously,$\lambda x_n \longrightarrow \lambda x$, $\lambda y_n \longrightarrow \lambda y$, and $\{\lambda x_n\} \in \lambda A$, $\{\lambda y_n\} \notin \lambda A$. Then we have $\lambda x \in \partial(\lambda A)$, i.e.$\partial(\lambda A) \supseteq \lambda\partial A$. hence $\partial A \supseteq \partial(\lambda A)/\lambda$, whence $\partial(\lambda A) \subseteq \lambda\partial A$. Obviously $\partial(\lambda A) = \lambda\partial A$.

Lemma 9 *Let X be a locally convex space, f a continuous convex functional on X satisfying (F_1), $0=f(0)<f(x)$ $(x \neq 0)$ and A a bounded closed circled convex set, $0 \in IntA$. Then $S_X \subseteq A$ if and only if $f(x) \geq 1$ for any $x \in \partial A$.*

Proof Necessity, suppose $S_X \subseteq A$. If there exists an $x \in \partial A$ such that $f(x)<1$, obviously, $x \in IntS_X$ is in contradiction with $S_X \subseteq A$. So for any $x \in \partial A$, we have $f(x) \geq 1$.

Sufficiency, suppose $0<f(x)<1$, $0 \in IntA$. There exist $0<\lambda<1$ such that $\lambda x \in A$. Putting $\lambda_0 = \sup\{0<\lambda\leq 1, \lambda x \in A\}$, by closure of A, we can know $\lambda_0 x \in A$. If $\lambda_0=1$, $x \in A$. If $0<\lambda_0<1$, since there exist $\lambda_n>\lambda_0$ such that $\lambda_n \longrightarrow \lambda_0$, $\lambda_n x \notin A$, we have $\lambda_0 x \in \partial A$.It follows that $f(\lambda_0 x) = \psi(\lambda_0)f(x) < 1$, which leads to a contradiction. Then we have $\{x: f(x)<1\} \subseteq A$, whence, $\overline{S_X} \subseteq A$

Lemma 10 *X, f and A are same to that of Lemma 9, and there exists a $\lambda_0>0$ such that $\lambda_0 S_X \subseteq A$. Then we have*

$$\sup_{\lambda>0,\lambda S_X\subseteq A}\psi(\lambda) = \inf_{x\in\partial A} f(x).$$

Proof Let $\lambda>0$ such that $\lambda S_X \subseteq A$, whence $S_X \subseteq \frac{1}{\lambda}A$. By Lemma 9, for any $x \in \partial A$, we have $f(x/\lambda) \geq 1$, whence $f(x) \geq \psi(\lambda)$. Then we have

$$\sup_{\lambda>0,\lambda S_X\subseteq A}\psi(\lambda) \leq \inf_{x\in\partial A} f(x).$$

On the other hand, putting $\inf_{x\in\partial A} f(x) = \rho > 0$, for any $0<\varepsilon<\rho$, there exists an $x \in \partial A$ such that

$$f(x) - \varepsilon \leq \inf_{y\in\partial A} f(y).$$

There exists a $\bar{\lambda} > 0$ such that $\psi(\bar{\lambda}) = f(x) - \varepsilon$. So for any $y \in \partial A$, we have $\psi(\bar{\lambda}) \leq f(y)$,whence $f(y/\bar{\lambda}) \geq 1$. Then for any $z \in \partial(\frac{1}{\bar{\lambda}}A)$, we have $f(z) \geq 1$. By Lemma 9, we have $\bar{\lambda} S_X \subseteq A$, hence

$$\sup_{\lambda > 0, \lambda S_X \subseteq A} \psi(\lambda) \geq \psi(\bar{\lambda}) = f(x) - \varepsilon \geq \inf_{y \in \partial A} f(y) - \varepsilon,$$

which, together with the opposition inequality, complete the proof of Lemma 10.

Noting $\inf_{x \in \partial A} f(x)$ in Lemma 10 as $\psi(r(A))$, we have

$$r(A) S_X \subseteq A.$$

Remark When X is a finite dimension space, the inf of $\inf_{x \in \partial A} f(x)$ in Lemma 10 can be obtained, hence $\partial(r(A)S_X) \cap \partial A \neq \emptyset$, and given $0 \in IntA$, there must exist $\lambda_0 > 0$ such that $\lambda_0 S_X \subseteq A$.

Theorem 11 *Let X_{n+1} be an $n+1$-dimensional locally convex space, f a continuous convex functional on X_{n+1} satisfying (F_1), $0 = f(0) < f(x)$ $(x \neq 0)$, and A a bounded closed circled convex set such that $0 \in \mathrm{Int} A$. Then we have*

$$d_n(A, X_{n+1}) = \psi(r(A)).$$

Proof By the remark of Lemma 10, there exists an $x \in \partial(r(A)S_X) \cap \partial A$. Since $x \in \partial A$, there exists a functional $\varphi \in X_{n+1}^* \setminus \{0\}$ such that

$$\varphi(x) = \sup_{y \in A} \varphi(y).$$

Put $\sup_{f(x) \neq 0} \frac{\psi(|\varphi(x)|)}{f(x)} = \beta$, and $G = \{z \in X_{n+1} : \varphi(z) = 0\}$, obviously, G is a hyperplane. By Lemma 8, we have

$$\rho_f(x, G) = \sup_{y \in A} \rho_f(y, G) = \sup_{y \in A} \inf_{g \in G} f(y - g),$$

hence

$$\beta f(x) \geq \psi(|\varphi(x)|) = \sup_{y \in A} \psi(|\varphi(y)|) \geq \sup_{y \in r(A) S_{X_{n+1}}} \psi(|\varphi(y)|) = \psi(r(A)) = \beta f(x),$$

hence $\rho_f(x, G) = \psi(r(A)) = \sup_{y \in A} \inf_{g \in G} f(y - g)$, then we have

$$d_n(A, X_{n+1}) \leq \psi(r(A)).$$

On the other hand, for any n-dimensional subspace G', there exists a $y \in X_{n+1} \setminus \{0\}$ such that $y \perp G'$. We have

$$\psi(r(A)) = \sup_{y \in r(A) S_{X_{n+1}}} \inf_{g' \in G'} f(y - g').$$

Since n-dimensional subspace G' is arbitrary, we have

$$d_n(A, X_{n+1}) \geq \psi(r(A)).$$

This, together with the opposite inequality shown above, proves the theorem.

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局部凸空间中的 f- 宽度

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摘 要: 本文将赋范空间中的宽度推广到了局部凸空间, 并得到了一些相应的结论.

关键词: f-n- 宽度; S_x.