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Multivariate Vector Valued Salzer's Theorem *

CHEN Zhi-bing
(Dept. of Math., Normal College, Shenzhen University, Guangdong 518060, China)

Abstract: In this paper, the important Salzer's theorem for rational interpolation is generalized to the multivariate vector valued case.

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1. Introduction

H.E.Salzer gave the following theorem in 1962 which can be used to convert nonlinear interpolation problem into the linear one in osculatory rational interpolation.

Theorem 1[1] *Denote by $H(x)$ the collection of all univariate polynomials, suppose $p(x), q(x) \in H(x), q(x_i) \neq 0, s_i \in N$. Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(\frac{p(x)}{q(x)}\right)_{x=x_i} = f^{(k)}(x_i) \Longleftrightarrow p^{(k)}(x_i) = \left(f(x)q(x)\right)^{(k)}(x_i), \quad k = 0, 1, \cdots s_i - 1. \quad (1)$$

Zhu and Gu ([2]), Gu ([3]) generalized Theorem 1 to the vector and matrix cases, respectively. And in this paper, Theorem 1 is further generalized to the bivariate vector case.

All vectors discussed in this paper are supposed to be d-dimensional, and $\vec{a}(x,y) = (a_1(x,y), a_2(x,y), \cdots, a_d(x,y))$ is called bivariate vector valued function.

Definition 1 *Let $P = (x,y), P_0 = (x_0, y_0), \vec{a}(x,y)$ has the limits $\vec{a} = (a_1, a_2, \cdots, a_d)$ as $P \to P_0$, if $\lim_{P\to P_0} a_i(x,y) = a_i$ for $i = 1, 2, \cdots, d$. $\vec{a}(x,y)$ is continuous at P_0 if $\lim_{P\to P_0} \vec{a}(x,y) = \vec{a}(x_0, y_0)$.*

Definition 2 *If $\lim_{\Delta x\to 0} \frac{1}{\Delta x}\{\vec{a}(x+\Delta x, y) - \vec{a}(x,y)\}$ exists, it is called the partial derivative*

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Biography: CHEN Zhi-bing (1968-), male, Ph.D., Associate Professor.

of $\vec{a}(x,y)$ with respect x at (x,y), and denote its value by $\frac{\partial\vec{a}(x,y)}{\partial x}$. Similarly, denote the partial derivative of $\vec{a}(x,y)$ with respect y at (x,y) by $\frac{\partial\vec{a}(x,y)}{\partial y}$.

nth partial derivatives of $\vec{a}(x,y)$ are defined as $(n-1)$st partial derivatives of $\vec{a}(x,y)$ and we denote it by $\frac{\partial^n\vec{a}(x,y)}{\partial x^u\partial y^v}$, $u+v=n$, u,v are nonegtive integers here and in the sequel of this paper.

2. Bivariate vector valued Salzer's theorem

Denote by $\vec{V}_d(x,y)$ the collection of all bivariate vector valued functions and by $\pi(x,y)$ the collection of all bivariate polynomials. Suppose all partial derivatives in this paper are continuous. The following lemma is obviously.

Lemma 1

$$\frac{\partial^k}{\partial x^u\partial y^v}\vec{a}(x,y)=\left(\frac{\partial^k}{\partial x^u\partial y^v}a_1(x,y),\frac{\partial^k}{\partial x^u\partial y^v}a_2(x,y),\cdots,\frac{\partial^k}{\partial x^u\partial y^v}a_d(x,y)\right),\ u+v=k.$$

By Leibnitz's formula and

$$\frac{\partial^k}{\partial x^u\partial y^v}\vec{a}(x,y)=\frac{\partial^v}{\partial y^v}\left(\frac{\partial^u}{\partial x^u}\vec{a}(x,y)\right),\ u+v=k, \tag{2}$$

it is easy to prove

Lemma 2 *Suppose $\vec{f}(x,y)\in\vec{V}_d(x,y), g(x,y)\in\pi(x,y)$ and their first nth partial derivatives are continuous. Then*

$$\frac{\partial^n}{\partial x^n}(\vec{f}g)=\sum_{i=0}^{n}\binom{n}{i}\frac{\partial^i\vec{f}}{\partial x^i}\frac{\partial^{n-i}g}{\partial x^{n-i}}, \tag{3}$$

$$\frac{\partial^n}{\partial y^n}(\vec{f}g)=\sum_{i=0}^{n}\binom{n}{i}\frac{\partial^i\vec{f}}{\partial y^i}\frac{\partial^{n-i}g}{\partial y^{n-i}}, \tag{4}$$

$$\frac{\partial^n}{\partial x^u\partial y^v}(\vec{f}g)=\sum_{i=0}^{u}\sum_{j=0}^{v}\binom{u}{i}\binom{v}{j}\frac{\partial^{i+j}\vec{f}}{\partial x^i\partial y^j}\frac{\partial^{u-i+v-j}g}{\partial x^{u-i}\partial y^{v-j}},\ u+v=n. \tag{5}$$

Theorem 2 *Suppose $\vec{P}(x,y)=(p_1(x,y),p_2(x,y),\cdots p_d(x,y))\in\vec{V}_d(x,y)$, $p_i(x,y)\in\pi(x,y)$, $i=1,2,\cdots d$, $Q(x,y)\in\pi(x,y)$ and $Q(x_i,y_j)\neq 0$, $\vec{f}(x,y)$ has first $(s_{ij}-1)$st continuous partial derivatives at (x_i,y_j). Then*

$$\frac{\partial^k}{\partial x^u\partial y^v}\frac{\vec{P}(x_i,y_j)}{Q(x_i,y_j)}=\frac{\partial^k}{\partial x^u\partial y^v}\vec{f}(x_i,y_j),\ k=0,1,\cdots,s_{ij}-1,\ u+v=k. \tag{6}$$

if and only if

$$\frac{\partial^k}{\partial x^u\partial y^v}\vec{P}(x_i,y_j)=\frac{\partial^k}{\partial x^u\partial y^v}(\vec{f}Q)(x_i,y_j),\ k=0,1,\cdots,s_{ij}-1,\ u+v=k. \tag{7}$$

Proof For simplicity, denote (6), (7) by

$$\frac{\partial^k}{\partial x^u \partial y^v}\frac{\vec{P}}{Q} = \frac{\partial^k}{\partial x^u \partial y^v}\vec{f} \tag{8}$$

and

$$\frac{\partial^k}{\partial x^u \partial y^v}\vec{P} = \frac{\partial^k}{\partial x^u \partial y^v}(\vec{f}Q) \tag{9}$$

respectively. It is easy to examine that theorem 2 is valid for $k = 0, 1$. Suppose theorem 2 is right for $k = 0, 1, \cdots, m-1$, i.e.

$$\frac{\partial^k}{\partial x^u \partial y^v}\frac{\vec{P}}{Q} = \frac{\partial^k}{\partial x^u \partial y^v}\vec{f}, \quad k = 0, 1, \cdots, m-1, \ u+v=k. \tag{10}$$

are equivalent to

$$\frac{\partial^k}{\partial x^u \partial y^v}\vec{P} = \frac{\partial^k}{\partial x^u \partial y^v}(\vec{f}Q), \quad k = 0, 1, \cdots, m-1, \ u+v=k, \tag{11}$$

we now prove that

$$\frac{\partial^k}{\partial x^u \partial y^v}\frac{\vec{P}}{Q} = \frac{\partial^k}{\partial x^u \partial y^v}\vec{f}, \quad k = 0, 1, \cdots, m, \ u+v=k, \tag{12}$$

if and only if

$$\frac{\partial^k}{\partial x^u \partial y^v}\vec{P} = \frac{\partial^k}{\partial x^u \partial y^v}(\vec{f}Q), \quad k = 0, 1, \cdots, m, \ u+v=k. \tag{13}$$

From (13) and (5), one can derive the following

$$\frac{\partial^m}{\partial x^u \partial y^v}\vec{P} = \frac{\partial^m}{\partial x^u \partial y^v}(\vec{f}Q),$$

$$\frac{\partial^m}{\partial x^u \partial y^v}(\frac{\vec{P}}{Q}Q) = \frac{\partial^m}{\partial x^u \partial y^v}(\vec{f}Q),$$

$$\sum_{i=0}^{u}\sum_{j=0}^{v}\binom{u}{i}\binom{v}{j}\frac{\partial^{i+j}}{\partial x^i \partial y^j}\frac{\vec{P}}{Q}\frac{\partial^{u-i+v-j}Q}{\partial x^{u-i}\partial y^{v-j}} = \sum_{i=0}^{u}\sum_{j=0}^{v}\binom{u}{i}\binom{v}{j}\frac{\partial^{i+j}\vec{f}}{\partial x^i \partial y^j}\frac{\partial^{u-i+v-j}Q}{\partial x^{u-i}\partial y^{v-j}},$$

$$u+v=m. \tag{14}$$

Since (10)is equivalent to (11), (14) can be simplified as

$$\frac{\partial^m}{\partial x^u \partial y^v}\frac{\vec{P}}{Q}Q = \frac{\partial^m}{\partial x^u \partial y^v}\vec{f}Q, \ u+v=m,$$

or

$$\frac{\partial^m}{\partial x^u \partial y^v}\frac{\vec{P}}{Q} = \frac{\partial^m}{\partial x^u \partial y^v}\vec{f}, \ u+v=m.$$

Hence one get (12).

On the other hand, form (12) and (6),

$$\frac{\partial^m}{\partial x^u \partial y^v}\vec{P} = \frac{\partial^m}{\partial x^u \partial y^v}(\frac{\vec{P}}{Q}Q) = \sum_{i=0}^{u}\sum_{j=0}^{v}\binom{u}{i}\binom{v}{j}\frac{\partial^{i+j}}{\partial x^i \partial y^j}\frac{\vec{P}}{Q}\frac{\partial^{u-i+v-j}Q}{\partial x^{u-i}\partial y^{v-j}}$$
$$= \sum_{i=0}^{u}\sum_{j=0}^{v}\binom{u}{i}\binom{v}{j}\frac{\partial^{i+j}}{\partial x^i \partial y^j}\vec{f}\frac{\partial^{u-i+v-j}Q}{\partial x^{u-i}\partial y^{v-j}} = \frac{\partial^m}{\partial x^u \partial y^v}(\vec{f}Q),$$

so one get (13). Therefor by induction the theorem is proved.

References:

[1] SALZER H E. *Note on osculatory rational interpolation* [J]. Math. Comp., 1962, **16**: 486-491.

[2] ZHU Gong-qin, GU Chuan-qing. *Salzer's theorem for vectors* [J]. J. Math. Res. Exposition, 1990, **4**: 516.

[3] GU Chuan-qing. *Salzer's theorem for matrices* [J]. J. Hefei Univ. Tech., 1993, **4**.

[4] WANG Ren-hong. *Numerical Rational Approximation* [M]. Shanghai: Shanghai Science and Technology Press, 1980.

Salzer 定理的二元向量形式

陈 之 兵

(深圳大学师范学院数学系，广东 深圳 518060)

摘 要: 本文将在切触有理插值中起重要作用的 Salzer 定理推广到了多元向量的情形.

关键词: 多元变量；有理插值； Salzer 定理.