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A Reexamination on the Problem of Advanced Response *

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Abstract: In the theory of signal analysis, the existence of the advanced response, whose appearance is earlier than the input, violates the law of causation. It is believed that the presence of advanced response is due to unreasonable setting of the characteristic of the system. Hereof, by means of nonstandard analysis, it is concluded that the advanced response arises from the course of Fourier analysis other than the transmission process of the signal.

Key words: nonstandard analysis theory; Fourier analysis; δ-function ; signal; advanced response.

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1. Introduction

As viewed from mathematics, a "signal" is a function of time and a "transmission system" is a rule for assigning to any a signal (input) another signal (response). It is quite obvious that the influence of a signal on a system should always lag behind its input. As a result, if $f(t)$ is a "causal signal", i. e., if $f(t) = 0$, if $t < 0$, then, the response of a system to $f(t)$, say $\bar{f}(t)$, must be also a causal signal:

$$\bar{f}(t) = 0, \quad \text{if } t < 0.$$

But, such is not always the case. Namely, there is response appearing earlier than its input. This phenomenon is called "advanced response", which violates law of causation.

It is believed that[4-11]:

(a) The presence of the advanced response is due to unreasonable setting of the characteristic of the system.

This argument is subsequently called in question.

2. Causal systems

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The response of a system, say L, to the input $f(t)$ is hereof written as $\bar{f}_L(t)$. In addition, only so called "linear systems" are considered. Such a system satisfies the condition that the spectrum function of the response is linear with that of the input. Namely, for

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\mathrm{d}t,$$

and

$$\bar{F}_L(\omega) = \int_{-\infty}^{+\infty} \bar{f}_L(t)e^{-i\omega t}\mathrm{d}t,$$

there exist real functions $K_L(\omega)$ and $\varphi_L(\omega)$ such that

$$\bar{F}(\omega) = K_L(\omega)e^{i\varphi_L(\omega)}F(\omega).$$

Here, $K_L(\omega)$ and $\varphi_L(\omega)$ are called the characteristic of the L.

Also, the characteristic of an arbitrary linear system is simply denoted as $K(\omega)$ and $\varphi(\omega)$. As such, while "Unit Step signal"

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

passing through an arbitrary linear system, the response is denoted as

$$\bar{H}(t) = \frac{K(0)}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{i\omega}K(\omega)e^{i(\omega t+\varphi(\omega))}\mathrm{d}\omega. \tag{1}$$

Let A be an ideal low-pass filter, whose characteristic is (where $K_\circ$, $\omega_\circ$ and $t_\circ$ are constants)

$$K_A(\omega) = \begin{cases} K_\circ, & \text{if } |\omega| \leq \omega_\circ, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi_A(\omega) = -\omega t_\circ.$$

Then, eq. (1) gives

$$\bar{H}_A(t) = \frac{K_\circ}{2} + \frac{K_\circ}{\pi}\int_0^{\omega_\circ} \frac{\sin\omega(t-t_\circ)}{\omega}\mathrm{d}\omega.$$

Clearly, $H(t)$ is a causal signal, but $\bar{H}_A(t)$ is not. This is an example of "advanced response".

According to the proposition (a), a system can be realized only if it does not bring about advanced response. Namely, a causal input passing through a realizable system turns into a causal response. Specially, the response of a realizable system to the $H(t)$ must be causal. By eq. (1), this condition is expressed as

$$\frac{K(0)}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{i\omega}K(\omega)e^{i(\omega t+\varphi(\omega))}\mathrm{d}\omega = 0, \quad \text{if } t < 0.$$

Taking the derivative on both sides, it is obtained that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} K(\omega)e^{i(\omega t+\varphi(\omega))}\mathrm{d}\omega = 0, \quad \text{if } t < 0.$$

According to the definition of the negative frequency, the above condition can be written as

$$\frac{1}{\pi}\int_{0}^{+\infty} K(\omega)\cos(\omega t+\varphi(\omega))\mathrm{d}\omega = 0, \quad \text{if } t < 0. \tag{2}$$

As usual, a system meeting eq. (2) is called "causal system". So, from the (a) it is concluded that

(b) Only causal systems are realizable.

As such, the appearance of the advanced response of the system A is due to that it is not a causal system, i. e., due to that $K_A(\omega)$ and $\varphi_A(\omega)$ fails to satisfy condition (2). This is the only existent explanation of advanced response. Nevertheless, this explanation is unsatisfactory.

Indeed, the A is an ideal system. But an ideal system never means a system violating causality. The "mass point" in Newton mechanics and the "continuum" in electromagnetic theory are also ideal systems, but Newton mechanics or electromagnetic theory never violates causality for this reason.

Specially, while passing through a system, say B, with the characteristic

$$K_B(\omega) = e^{-\omega^2}, \quad \varphi_B(\omega) = 0,$$

the $H(t)$ is transformed into

$$\bar{H}_B(t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{\frac{t}{2}} e^{-x^2}\mathrm{d}x,$$

which is also a non-causal signal, so that the B is not a causal system either. But, $K_B(\omega)$ and $\varphi_B(\omega)$ are much similar to the characteristic of a resistance coupling enlarger. It is hard to imagine that such a characteristic is associated with the violation of causality.

Nor is this all, there exists a more sharp contradiction in this explanation.

3. Virtual systems

If a signal does not pass through any system, it may be regarded as passing through a system, say C, with the characteristic

$$K_C(\omega) = 1, \quad \varphi_C(\omega) = 0.$$

the C is a "virtual system", which is undoubtedly realizable. On the other hand, if the C is a causal system, eq. (2) gives

$$\frac{1}{\pi}\int_0^{+\infty} \cos\omega t\mathrm{d}\omega = \lim_{\omega\to+\infty}\frac{\sin\omega t}{\pi t} = 0, \quad \text{if } t < 0.$$

But, for a given $t < 0$, when $\omega \to +\infty$, the $\frac{\sin \omega t}{\pi t}$ vibrates over $[-\frac{1}{\pi t}, \frac{1}{\pi t}]$ and does not approach zero as a limit. Therefore, *the C is realizable but is not a causal system.* This is a reverse example to the (b).

When we consider another signal, this contradiction presents a more puzzling appearance. Just like the "mass point" in mechanics, there is an abstract called "impulse signal" in signal analysis theory. As a mass point is a body concentrated into a point, so is an impulse signal a signal concentrated into a moment. Both of them can be expressed by a δ-function (Dirac delta function). The earliest definition of δ-function is given by

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

and

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1.$$

If $\varepsilon > 0$ is small enough, the function

$$\gamma_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } 0 \le t < \varepsilon, \\ 0, & \text{otherwise} \end{cases}$$

meets the above definition approximately, so that it is believed that

$$\delta(t) = \lim_{\varepsilon \to +0} \gamma_\varepsilon(t). \tag{4}$$

By means of δ-function , any impulse signal can be expressed as $q\delta(t-\tau)$, where q and τ are constants. Specially, the signal denoted by eq. (4) is called "unit impulse".

It is well known that for any a continuous function $f(t)$, there exists $\tau \in (0, \varepsilon)$, such that

$$\int_{-\infty}^{+\infty} \gamma_\varepsilon(t) f(t)\mathrm{d}t = \frac{1}{\varepsilon}\int_0^\varepsilon f(t)\mathrm{d}t = f(\tau).$$

Let $\varepsilon \to +0$, it is obtained that

$$\int_{-\infty}^{+\infty} \delta(t) f(t)\mathrm{d}t = f(0).$$

This formula is usually called "screening property" of δ-function . When $f(t) = e^{-i\omega t}$, it gives

$$\int_{-\infty}^{+\infty} \delta(t) e^{-i\omega t}\mathrm{d}t = 1.$$

This formula shows that the spectrum function of $\delta(t)$ is 1. Therefore, the response to the unit impulse, which is usually called "impulse response", of a system is

$$\bar{\delta}(t) = \frac{1}{\pi}\int_0^{+\infty} K(\omega)\cos(\omega t + \varphi(\omega))\mathrm{d}\omega. \tag{5}$$

Which is usually called "impulse response". Comparing eq. (5) with eq. (2) it is seen that (b) means *the impulse response of a realizable system must be causal.* Correspondingly, the advanced response phenomenon manifests itself by the fact that the impulse response of a certain system is non-causal. For example, the impulse response of the A is

$$\bar{\delta}_A(t) = \frac{K_o \sin\omega(t - t_o)}{\pi(t - t_o)};$$

and that of the B is

$$\bar{\delta}_B(t) = \frac{1}{2\sqrt{\pi}} e^{-\frac{t^2}{4}}.$$

Both of them are non-causal signal.

To return to virtual system C, of which the impulse response is

$$\bar{\delta}_C(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\mathrm{d}\omega = \lim_{\omega\to+\infty}\frac{\sin\omega t}{\pi t}.$$

Yet, for the virtual system, the response of the $\delta(t)$ should be equivalent to the input, namely, $\bar{\delta}_C(t) = \delta(t)$. So,

$$\delta(t) = \lim_{\omega\to+\infty}\frac{\sin\omega t}{\pi t}. \tag{6}$$

This formula is called "Fourier core", which can be found in a more direct way: Since that the spectrum function of $\delta(t)$ is 1, the Fourier development of $\delta(t)$ is expressed as eq. (6).

Fourier core is well known by physicists. It has given a lot of correct results, but it leads to a contradiction here: From eq. (4), we see that $\delta(t)$ is a causal signal, but eq. (6) indicates that $\delta(t)$ is a non-causal signal. To avoid this contradiction, it must be recognized that eq. (4) and eq. (6) denote two distinct δ-functions. If so, however, another contradiction arises: A unit impulse passing through a virtual system is no longer a unit impulse; it is transformed into a Fourier core. But passing through a virtual system means through no system. So, in this process, *the unit impulse from a causal signal changes into a non-causal signal without any reason.*

The argumentation above leading to various contradictions is expressed by the language of physical intuition. In the view of classical mathematics, this language makes no sense in some respects. For example, $+\infty$ is not a number, so that eq. (3) is meaningless. Similarly, when $\omega \to +\infty$, the $\frac{\sin\omega t}{\pi t}$ has no limit, thereby eq. (6) is also meaningless, and so on.

All of these formulae and reasonings are the products of the period in which the understandings on δ-function are in confusion. Today, a better comprehension of δ-function has achieved, can we get things into shape from the above mess?

In Distribution theory, starting from the screening property, δ-function is defined as a linear functional. As a result, the question whether or not an impulse is a causal signal is meaningless. Thus, the confused language disappears; but, at the same time, the physical substance expressed by the confused language also disappears. So, Distribution theory is inapplicable for resolving our problem.

What we need here is such a mathematics theory: It is as strict as Distribution theory but it is sufficiently intuitive to manifest the facts such as "unit impulse is causal and

Fourier core is not". It will be seen that non-standard analysis[1-3] is just such a theory.

4. Advanced response to impulse signal

In what follows, a δ-function theory based on non-standard analysis [2] is applied.

Just like in Distribution theory, in the nonstandard analysis a δ-function is also defined by means of the screening property.

Definition *If $\mu(t)$ is a function from $^*R$ to $^*R$, and there exists a standard function $g(t)$ meeting $\int_{-\infty}^{+\infty} g(t)\mathrm{d}t = 1$ such that $\mu(t) = {}^*g(t)$, and the equation*

$$\mathrm{st}^* \int_{^*-\infty}^{^*+\infty} {}^*f(t)\mu(t)\mathrm{d}t = f(0)$$

is true for any a standard function $f(t)$ that meets a given condition E, then, the $\mu(t)$ is called a "δ-function with regard to E".

Now, two terms are introduced here: The collection of objects described by the infinitesimal elements in nonstandard analysis is called "micro world", and that described by the finite numbers, "macro world".

In the macro world, the unit impulse can be conceived as a rectangular signal that is narrow enough. This ambiguous idea is now expressed accurately as

$$\delta_n(t) = \begin{cases} n, & \text{if } 0 \le t < \frac{1}{n}, \\ 0, & \text{otherwise,} \end{cases}$$

where $n \in {}^*\mathrm{N} - \mathrm{N}$. Clearly, for any a standard function $f(t)$ which is continuous at $t = 0$,

$$\mathrm{st}^* \int_{^*-\infty}^{^*+\infty} {}^*f(t)\delta_n(t)\mathrm{d}t = f(0).$$

So, $\delta_n(t)$ is a δ-function with regard to the condition "continuous at $t = 0$". Here, $\delta_n(t)$ is a formal expression to eq. (4) and is also called "unit impulse".

On the other hand, if $f(t)$ is a function from R to R, continuous and integrable in the R, then, for any $\Omega \in {}^*\mathrm{N} - \mathrm{N}$,

$$\mathrm{st}^* \int_{^*-\infty}^{^*+\infty} {}^*f(t)\frac{\sin \Omega t}{\pi t}\mathrm{d}t = f(0).$$

So, the nonstandard function $\Delta_\Omega(t) = \frac{\sin \Omega t}{\pi t}$ is a δ-function with regard to the condition "continuous and integrable in the R". It is a formal expression to eq. (6) and is also called "Fourier core".

Here, $\delta_n(t)$ is a causal signal, but $\Delta_\Omega(t)$ is not.

Assume that a δ-function , say $\mu(t)$, causal or non-causal, passes through a system. Then, *as viewed from an observer in macro world*, since that the spectrum function of the δ-function is 1, the response to $\mu(t)$ is

$$\bar{\mu}(t) = \frac{1}{\pi}\int_0^{+\infty} K(t)\cos(\omega t + \varphi(t))\mathrm{d}\omega.$$

As we know, $\bar{\mu}(t)$ is a non-causal signal generally. When $\mu(t)$ is the non-causal signal, say $\Delta_\Omega(t)$, this result is natural; but when $\mu(t)$ is a causal signal, say $\delta_n(t)$, the very outcome means advanced response. Why is it that?

As we see, the calculation of the response is based on Fourier analysis, any input $f(t)$ is regarded as the Fourier development of the $f(t)$; also, regarding the spectrum function of $\mu(t)$ as 1 means regarding the Fourier development of $\mu(t)$ as $\Delta_\Omega(t)$. Therefore, the $\mu(t)$ as an input signal is regarded as $\Delta_\Omega(t)$. Consequently, so-called "causal system" is a system meeting the condition that *the response of it to the non-causal signal $\Delta_\Omega(t)$ is causal*. For a system, that is an unreasonable requirement. Therefore, the (b) is a false proposition. As a result, it is no longer a contradiction that the virtual system is not a causal system but still realizable.

Now, the reason why the response to $\delta_n(t)$ of the system A, B or C is a non-causal signal can be described roughly as follows:

Firstly, in the course of Fourier development, the causal signal $\delta_n(t)$ is substituted by the non-causal signal $\Delta_\Omega(t)$. Secondly, while passing through A or B, the non-causal signal $\Delta_\Omega(t)$ is transformed into another non-causal signal $\bar{\delta}_A(t)$ or $\bar{\delta}_B(t)$ respectively; while through a virtual system C, it remains $\Delta_\Omega(t)$.

It is thus seen that the advanced response of unit impulse arises from the course of Fourier development other then the transmission process.

5. Advanced response to step signal

In a sense, nonstandard analysis is a formal manifestation of the intuitive language for infinitesimal given by Leibniz. Expressed by Leibniz's language, the production mechanism of advanced response for unit step signal $H(t)$ can be described as follows.

When the signal $H(t)$ goes into a system, it is naturally divided into a set of infinitesimal impulses in time sequence, each of them are causal signal. While performing Fourier development for $H(t)$, we unconsciously substitute each infinitesimal impulse by an infinitesimal Fourier core, a non-causal signal. But the composite of these cores is still the causal signal $H(t)$. Now, the advanced response has been produced but it is hidden from view. In the transmission process, every core is endowed with a response, another infinitesimal non-causal signal. Correspondingly, the $H(t)$ changes into another signal $\bar{H}(t)$ which is composed of those non-causal responses. This $\bar{H}(t)$ is no longer a causal signal. As a result, the advanced response appears. In what follows, this process will be expressed by modern mathematical language.

To examine the transmission of $H(t)$ in the micro world, a micro model of $H(t)$, namely, a nonstandard function whose standard part is $H(t)$ must be given. In the macro world, $H(t)$ is actually regarded as an idealization of rectangular signal, which is sufficiently wide. So, It is quite natural to take

$$I_\lambda(t) = \begin{cases} 1, & \text{if } 0 \le t < \lambda, \\ 0, & \text{otherwise} \end{cases}$$

and $\lambda \in {}^*\mathrm{N} - \mathrm{N}$ as the micro model of $H(t)$. Introducing $m \in {}^*\mathrm{N} - \mathrm{N}$ and

$$S_m(t) = \begin{cases} 1, & \text{if } 0 \le t < \frac{\lambda}{m}, \\ 0, & \text{otherwise,} \end{cases}$$

a division of $I_\lambda(t)$ in time sequence is expressed as

$$I_\lambda(t) = {}^*\sum_{k=0}^{m-1} S_m(t - \frac{k}{m}\lambda). \tag{7}$$

Supposing that $n \in {}^*\mathrm{N} - \mathrm{N}$ and $m = n\lambda$, the definition of $\delta_n(t)$ gives,

$$nS_{n\lambda}(t) = \delta_n(t).$$

So, eq. (7) is rewritten as

$$I_\lambda(t) = {}^*\sum_{k=0}^{n\lambda-1} \frac{1}{n}\delta_n(t - \frac{k}{n}).$$

As we know, accompanying the course Fourier development of $H(t)$ in the macro world, $\delta_n(t)$ is substituted by $\Delta_\Omega(t)$, and thereby each causal signal $\delta_n(t - \frac{k}{n})$ is substituted by a non-causal signal $\Delta_\Omega(t - \frac{k}{n})$. Correspondingly, $I_\lambda(t)$ is substituted by

$$J_\lambda(t) = {}^*\sum_{k=0}^{n\lambda-1} \frac{1}{n}\Delta_\Omega(t - \frac{k}{n}).$$

$I_\lambda(t)$ as the composite of causal signals $\delta_n(t - \frac{k}{n})$ is also a causal signal, while $J_\lambda(t)$ as that of non-causal signals $\Delta_\Omega(t - \frac{k}{n})$ is also a non-causal signal. But its standard part is still a causal signal:

$$\mathrm{st} J_\lambda(t) = 0, \quad \text{if } t < 0.$$

So, though the causal signal $I_\lambda(t)$ has been substituted by a non-causal signal $J_\lambda(t)$ in the course of Fourier development, this effect does not emerge in the macro world.

Now, consider the transmission process. In this process, each non-causal signal $\Delta_\Omega(t - \frac{k}{n})$ is transformed into another non-causal signal $\bar{\Delta}_\Omega(t - \frac{k}{n})$, and correspondingly the non-causal signal $J_\lambda(t)$ is transformed into another non-causal signal

$$\bar{J}_\lambda(t) = {}^*\sum_{k=0}^{n\lambda-1} \bar{\Delta}_\Omega(t - \frac{k}{n}),$$

of which the standard part is no longer a causal signal, so that the effect of advanced response appears in the macro world.

Therefore, as viewed from an observer of macro world, it seems that the advanced response of $H(t)$ emerges in the transmission process. But when we gaze at the transmission of the infinitesimal elements of the inputting signal, it is easy to find that it actually arises from the course of Fourier development.

6. Conclusion

To sum up, it has been demonstrated that the advanced response originates from Fourier analysis instead of the unreasonable setting of the characteristic of the transmission system. As a result, the argumentation base on the (a), which forms a great subject in the theory of signal analysis, is wrong from the very beginning. From this instance, it is seen that standard analysis as a mathematics means of the experimental sciences is not complete. For this reason, we have been mistaken about some problems. Merely to correct these mistakes, nonstandard analysis is indispensable.

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对提早响应问题的再考察

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摘 要: 本文将非标准分析应用于信号分析中的提早响应问题，证明提早响应产生于付里叶分析过程而不是产生于信号传输过程．这一例子表明标准分析对于实证科学是不完备的．