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Some Remarks for Discrete Versions of Nodal Domain Theorems *

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Abstract: In this paper, an error is firstly pointed out in the proof of the main theorems (Theorem 4 and Theorem 6) in [1]. Then the error is corrected and the right proof is given.

Key words: nodal domain theorem; discrete version.

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1. Introduction

The nodal domain theorems play important roles in the researches on eigenvalues as well as eigenvectors of operators and matrices. On the continuous cases, this question has been addressed by work of R.Courant and D.hilbert[2], in which the continuous nodal domain theorems were shown. Then A.Pleijlel[3] proposed the discrete version of nodal domain theorems that are Theorem 4 and Theorem 6 in [1]. Recently, Art M. Duval and Victor Reiner proved the two theorems by linear algebra method in [1]. In their proofs, a key calculational lemma, Lemma 5, was used. But the lemma is wrong, the proof of Theorem 4 and Theorem 6 are also wrong. In this paper, we shall show that Lemma 5 of [1] is wrong by an example at first. Then we correct the error of Art M. Duval and Victor Reiner's proofs and give the right proof.

We first give some terminology. Let L denote a symmetric $n \times n$ matrix with real entries. We can associate to L a graph having vertex set $V = \{1, 2, \cdots, n\}$, and edge set $E = \{vw | L_{vw}(= L_{vw}) \neq 0\}$. L is said to be indecompasible if $G(L)$ is connected. Let set $R^V = \{g | g : V \to R\}$, then given a vector $f \in R^n$, we will also think of f as an element of R^v, that is, as a function $f : V \to R$. We define the nodal domains V_1, V_2, $\cdots$ of f with respect to L to be the vertex sets of the connected components in the graph $G'(L) =$

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(V', E'), where $V' = \{v | v \in V, f(v) \neq 0\}$, $E' = \{vw | vw \in E; v, w \in V'; f(v)f(w) > 0\}$.

2. Results

Lemma 5 in [1] is the following:

Let L be a symmetric matrix with rows and columns indexed by a set V, and let $f \in R^v$. If $V_1, V_2, \cdots, V_{s+1}$ are subsets of V and one defines for $i = 1, 2, \cdots, s+1$ functions

$$f^{(i)}(v) = \begin{cases} f(v), & \text{if } v \in V_i, \\ 0, & \text{else.} \end{cases}$$

Non-zero function $\varphi = \sum_{i=1}^{s+1} c_i f^{(i)}$, $(c_1, c_2, \cdots, c_{s+1}$ are not all 0), then for any $\lambda \in R$, one has

$$\begin{aligned}\langle \varphi, L\varphi \rangle - \lambda \langle \varphi, \varphi \rangle = &\sum_i c_i^2 \sum_{v \in V_i} (f(v)(Lf)(v) - \lambda f(v)^2) - \\ &\sum_{i<j} (c_i - c_j)^2 \sum_{(v,w) \in V_i \times V_j} L_{vw} f(v) f(w).\end{aligned}$$

In the proof of [1], an equation

$$\sum_{w \in V_i} L_{vw} f^{(i)}(w) = (Lf)(v) - \sum_{j \neq i} \sum_{w \in V_j} L_{vw} f(w), \quad v \in V_i,$$

was used. Let $\overline{V} = V - (V_1 \cup V_2 \cup \cdots \cup V_{s+1})$. If $f(\overline{v}) \neq 0$ for some $\overline{v} \in \overline{V}$, then we can show that the equation is not hold by the following example. So in this case, the lemma is wrong. While Lemma 5 was used in the proofs of Theorem 4 and Theorem 6 ([1]) just in this case. Thus the proofs were also wrong.

Example Let

$$L = \begin{bmatrix} 2 & -1 & 1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 1 & -1 & 2 \end{bmatrix}, \quad V = \{1, 2, 3, 4, 5, 6\}.$$

Suppose that

$$V_1 = \{1, 3\}, V_2 = \{2\}, V_3 = \{4\}, V_4 = \{5\},$$

and

$$f = (2, 1, 1, -1, 1, -1), c_1 = 2, c_2 = 1, c_3 = -1, c_4 = -1.$$

So we have

$$f^{(1)} = (2, 0, 1, 0, 0, 0), f^{(2)} = (0, 1, 0, 0, 0, 0), f^{(3)} = (0, 0, 0, -1, 0, 0), f^{(4)} = (0, 0, 0, 0, 1, 0),$$

and

$$\varphi = (4, 1, 2, 1, -1, 0).$$

Then we shall get, for $\lambda = 1$, $\langle \varphi, L\varphi \rangle - \langle \varphi, \varphi \rangle = 27$, while

$$\sum_i c_i^2 \sum_{v \in V_i} (f(v)(Lf)(v) - f(v)^2) - \sum_{i<j} (c_i - c_j)^2 \sum_{(v,w) \in V_i \times V_j} L_{vw} f(v) f(w) = 37 \neq 27.$$

Now, let's pay our attention to Theorem 6 in [1] that was the following:

Let L be a real symmetric indecomposable matrix with non-positive off-diagonal entries, i.e. $L_{vw} \leq 0$ for $v \neq w$, and the eigenvalues of L are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, the corresponding eigenvectors are $\varphi_1, \varphi_2, \cdots, \varphi_n$. If $f \in R^V$ satisfies $(Lf/f) \leq \lambda_s$ pointwise, that is $(Lf)(v) \leq \lambda_s$ whenever $f(v) \neq 0$, then f has at most s nodal domains with respect to L.

In the proof of [1], "f has at most $s+1$ domains $V_1, V_2, \cdots, V_{s+1}$" was assumed at first, then let $\varphi = \sum_{i=1}^{s} c_i f^{(i)}$ and using Lemma 5 of [1] with $\lambda = \lambda_s$ deduced $\frac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \leq \lambda_s$.

By the definition of nodal domains, we know that for any $v \in V_{s+1}$, $f(v) \neq 0$, which means for some $\overline{v} \in \overline{V} = V - (V_1 \cup V_2 \cup \cdots \cup V_s)$, $f(v) \neq 0$ since $V_{s+1} \subseteq \overline{V}$. So Lemma 5 can not be used to deduce the result we desired. In fact, this result, $\frac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \leq \lambda_s$, can be deduced in another way. We compute

$$\begin{aligned}
\langle \varphi, L\varphi \rangle &= \sum_{i=1}^{s} \sum_{j=1}^{s} \langle c_i f^{(i)}, L c_j f^{(j)} \rangle \\
&= \sum_i c_i^2 \sum_{v \in V_i} f^{(i)}(v) \sum_{w \in V_i} L_{vw} f^{(i)}(w) + \sum_{(i,j), i \neq j} c_i c_j \sum_{v \in V_i} f^{(i)}(v) \sum_{w \in V_i} L_{vw} f^{(j)}(w) \\
&= \sum_i c_i^2 \sum_{v \in V_i} f(v) \left[(Lf)(v) - \sum_{i \neq j} \sum_{w \in V_j} L_{vw} f(w) - \sum_{w \in \overline{V}} L_{vw} f(w) \right] + \\
&\quad \sum_{(i,j), i \neq j} c_i c_j \sum_{v \in V_i} \sum_{w \in V_j} L_{vw} f(v) f(w) \\
&= \sum_i c_i^2 \sum_{v \in V_i} f(v)(Lf)(v) - \sum_{i<j} (c_i - c_j)^2 \sum_{(v,w) \in V_i \times V_j} L_{vw} f(v) f(w) - \\
&\quad \sum_i c_i^2 \sum_{v \in V_i} \sum_{w \in \overline{V}} L_{vw} f(v) f(w),
\end{aligned}$$

$\lambda_s \langle \varphi, \varphi \rangle = \sum_i c_i^2 \sum_{v \in V_i} \lambda_s f(v)^2$. So

$$\begin{aligned}
\langle \varphi, L\varphi \rangle - \lambda_s \langle \varphi, \varphi \rangle = &\sum_i c_i^2 \sum_{v \in V_i} (f(v)(Lf)(v) - \lambda_s f(v)^2) - \\
&\sum_{i<j} (c_i - c_j)^2 \sum_{(v,w) \in V_i \times V_j} L_{vw} f(v) f(w) - \\
&\sum_i c_i^2 \sum_{v \in V_i} \sum_{w \in \overline{V}} L_{vw} f(v) f(w). \qquad (1)
\end{aligned}$$

Multiplying the hypothesis $(Lf)(v)/f(v) \le \lambda_s$ for $f(v) \neq 0$ by $f(v)^2$ gives

$$f(v)(Lf)(v) - \lambda_s f(v)^2 \le 0.$$

So the first sum on the right-hand side of Eq. (1) is non-positive. By the assumption on L, i.e., $L_{vw} \le 0$ $(v \neq w)$, and the definition of nodal domains, we have

$$L_{vw} f(v) f(w) \ge 0.$$

So the second sum and the third sum are non-negative since $(c_i - c_j)^2 \ge 0$ and $c_i^2 \ge 0$. This gives

$$\langle \varphi, L\varphi \rangle - \lambda_s \langle \varphi, \varphi \rangle \le 0,$$

that is $\frac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \le \lambda_s$ as desired.

The right proof Theorem 4 can also be given with the same method. We will not discuss it here.

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关于离散形式结点域定理证明的几点注记

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摘 要: 本文指出了文 [1] 的主要定理 (定理 4 与定理 6) 证明中的错误，并对其进行了修正，给出了正确的证明.

关键词: 结点域定理；离散形式.