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A New Method for Computing the 1-Dimensional Homology Group of a Given 2-Complex *

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Abstract: In [3], a vector space associate with a graph G: "its cycle space" was described over the two element field $\mathbf{Z}_2$. Here we generalize the theory to the ring $\mathbf{Z}$ to compute 1-dimensional homology group of a given 2-complex with a combination of algebraic and graph-theoretic method.

Key words: 1-chain; 1-cycle; complex; homology group.

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1. Introduction

The general problem that we consider in this paper is to compute $H_1(K^n)$, the 1-dimensional homology group of a given n-complex K^n. Our approach is a combination of algebraic and graph-theoretic method. Let K^n be a given n-complex, $n \geq 2$, and K^r its r-skeleton. We first suggest a method for computing $H_1(K^1)$ $(= Z_1(K^1))$. Then we try to apply this result to compute $H_1(K^n)$, using the formula

$$H_1(K^n) \cong H_1(K^2) = H_1(K^1)/B_1(K^2). \tag{1}$$

Our algebraic terminology and notation follow Croom [2], and the graph-theoretic ones are essentially those of Bondy and Murty [1]. For the sake of completeness we include here our main notation. By $\mathbf{Z}$ we shall denote the group of integers, by $\mathbf{Q}$ the field of rational numbers. By $Z_1(K)$ we denote the 1-dimensional cycle group of K; it will be treated as a vector space rather than the group (see [2]).

In Section 2, we give a general outline for our method. It consists of five steps. Step 1 is developed in Sections 3 and 4, Step 2 in Section 5, Steps 3 and 4 in Section 6. We analyze

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the time complexity for our method in Section 8. In section 9, we illustrate our method, applying it to 2-pseudomanifolds. In the last Section, an open problem is presented.

2. The method

Here we shall describe the method for computing $H_1(K^n)$, the 1-dimensional homology group of a given n-complex K^n.

Input. K^n, a given n-complex.

Step 1. Determine a basis, $\{s_k\}$, of $Z_1(K^n)$.

Step 2. Partition the set $\{s_k\}$ into homology classes. Choose one representative in each of the classes and designate them by S_i, $i = 0, 1, \ldots, \mu$.

Step 3. Partition the set $\{S_i\}$ into at most three types of sets:

(i) It has only one 1-cycle which is homologous to zero.

(ii) Each element of this type, say S_{i_k}, satisfies $p_{i_k} \cdot S_{i_k} \sim 0$ (homologous to zero), where the integer $p_{i_k} \geq 2$.

(iii) Delete the elements of Types (i) and (ii) from $\{S_i\}_{i=1}^{\mu}$, the rest elements of $\{S_i\}_{i=1}^{\mu}$ consist of the elements of Type (iii).

Step 4. Find in Type (iii) a maximal subset of 1-cycles linear independent with respect to homology, denote its size by ν.

Output. The group $H_1(K^n) \cong \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{\nu} \oplus \mathbf{Z}_{p_1} \oplus \ldots \oplus \mathbf{Z}_{p_m}$.

3. 1-chains from the graph-theoretic point of view

By Eq.(1), we can assume $n = 2$ without lost of generality. Let K^2 be an oriented 2-complex. The 1-skeleton of K^2 will be denoted by $D = D(K)$; thus D is actually a digraph. Any 1-chain c_1 on K^2 can be written as a formal sum of elementary 1-chains

$$c_1 = \sum g_{ij} \cdot \langle a_i a_j \rangle, \tag{2}$$

where the sum is taken over the 1-simplexes $\langle a_i a_j \rangle$ of K^2, and $g_{ij} \in \mathbf{Z}$. Since $\partial(\langle a_i a_j \rangle) = 1 \cdot \langle a_j \rangle - 1 \cdot \langle a_i \rangle$ for an elementary 1-chain $\langle a_i a_j \rangle$ with orientation given by $a_i < a_j$, we have

$$\partial(c_1) = \sum g_{ij} \cdot (1 \cdot \langle a_j \rangle - 1 \cdot \langle a_i \rangle) = \sum h_i \cdot \langle a_i \rangle, \tag{3}$$

where the second sum is taken over the 0-simplexes of K^2, and $h_i \in \mathbf{Z}$.

With c_1 we associate a digraph $N = N(c_1)$ with a non-negative integer-valued function $f = f(c_1)$ defined on the arc set $A = A(N)$. For this, every arc $a_i a_j$ of N is assigned weight g_{ij}; in case $g_{ij} < 0$ we change it to $-g_{ij}$ with reversing the direction of the arc $a_i a_j$. The resulting arc-weighted digraph defines N. From Eqs.(3) one can see that N has the property $f_D^+(a) = f_D^-(a)$ ("conservation condition") for every vertex a not occurring in the second sum of Eqs.(3) ("intermediate vertex"). Thus, from the graph-theoretic point of view, a 1-chain is a network N with a flow f. Denote by X its source set, and by Y its sink set.

As an application, recall that the resultant flow out of X is equal to the resultant flow into Y ([1, Ex. 11.1.3]). This graph-theoretic fact implies that a multiple of a single vertex (i.e. a 0-simplex) cannot be the boundary of any 1-chain. The latter algebraic fact is implicitly used in [2] for proving [2, Theorem 2.4].

4. Algebraic 1-cycles and graph-theoretic cycles

Let $\tilde{D}$ denote the undirected graph obtained from D by omitting the directions of the edges. An oriented cycle $C = (a_{i_1}, a_{i_2}, \ldots, a_{i_n}, a_{i_1})$ of $\tilde{D}$ will be represented as either of the following formal sums of its edges

$$C = a_{i_1} a_{i_2} + a_{i_2} a_{i_3} + \ldots + a_{i_n} a_{i_1} = -a_{i_2} a_{i_1} - a_{i_3} a_{i_2} - \ldots - a_{i_1} a_{i_n}. \tag{4}$$

When operating with such formal sums, we shall suppose $a_i a_j = -a_j a_i$. A cycle vector of $\tilde{D}$ is a formal linear combination of its oriented cycles with integral coefficients. Cycle vectors $C_1, \ldots, C_k$ are called *linearly independent* if $\sum_{i=1}^{k} g_i \cdot C_i = 0$ ($g_i \in \mathbf{Z}$) implies $g_i = 0$, for each i. Denote by $S(\tilde{D})$ the *cycle space* of $\tilde{D}$, that is, the vector space of cycle vectors of $\tilde{D}$ over the field $\mathbf{Q}$.

Note: It is easy to see that the linear independence of cycle vectors over the ring $\mathbf{Z}$ is equivalent to their linear independence over the field $\mathbf{Q}$.

Since from the graph-theoretic point of view an algebraic 1-cycle is a network with $X = \emptyset$ and $Y = \emptyset$, algebraic 1-cycles on K^2 and cycle vectors of $\tilde{D}$ are identical:

Lemma 1 $Z_1(K^2) = S(\tilde{D})$.

Now we proceed to determine the basis of $S(\tilde{D})$. For this, first fix on a spanning tree, T, of $\tilde{D}$. Let $\{e_k\}$ be the set $E(\tilde{D}) \setminus E(T)$. Clearly, for each e_k, there is precisely one cycle of $\tilde{D}$ which contains e_k and has all other edges in T. That cycle determines two oriented cycles. Choose any of them and designate it by s_k.

Lemma 2 *The set* $\{s_k\}$ *forms a basis of* $S(\tilde{D})$.

Proof Let C be an arbitrary oriented cycle of D. Let $e_{k_1}, \ldots, e_{k_M}$ denote the edges of C which are not in T. We shall prove that

$$C = \sum_{i=1}^{M} \alpha_i s_{k_i}, \tag{5}$$

where α_i is 1 or -1 according to whether or not (respectively) the orientations of C and s_{k_i} are consistent on the edge e_{k_i}. To prove Eq.(5), replace e_{k_i}, for each i, by the (oriented) path $s_{k_i} - e_{k_i}$. We thus obtain a closed (oriented) walk of T. This walk traverses each of it's edges some even number of times, in alternating directions, because T is a tree. This immediately implies Eq.(5), which completes the proof. □

Lemmas 1 and 2 imply

Lemma 3 *The set* $\{s_k\}$ *forms a basis of* $Z_1(K^2)$.

5. Checking the homology of 1-cycles

Here we describe a method to verify whether or not two given 1-cycles z_1 and z_2 are homologous. This problem is equivalent to the problem of checking whether a given 1-

cycle z $(= z_1 - z_2)$ is homologous to zero or not. Our method is straightforward. Write a general 2-chain

$$c_2 = \sum_{i=1}^{\alpha_2} g_i \cdot \sigma_i^2, \tag{6}$$

where the sum is taken over all 2-simplexes of K. Compute $\partial(c_2)$. Then write the system of linear algebraic equations determined by the equation

$$z = \partial(c_2). \tag{7}$$

For this demand that the coefficient appearing with each 1-simplex σ^1 in the left-hand side of Eq. (7) is equal to the total coefficient with σ_i^1 in the right-hand side. The so-obtained system (7) has α_1 equations and α_2 unknowns (g_i). Then $z \sim 0$ if and only if the system (7) has an integer-valued solution.

6. Partitioning the set $\{S_i\}$

Partition $\{S_i\}$ obtained by the preceding step into at most three types of sets. Type (i) has only one 1-cycle, which is homology to zero, denoted by S_0, which can be determined in Step 2. In order to determine the elements of Type (ii), namely, whether there exists $p_{i_k} \geq 2$ such that $p_{i_k} \cdot S_{i_k} \sim 0$, we should consider the equation

$$p_{i_k} \cdot S_{i_k} = \partial(\sum_{i=1}^{\alpha_2} g_i \cdot \sigma_i^2), \tag{8}$$

and write the corresponding system of linear algebraic equation with $\alpha_2 + 1$ unknowns (g_i and p_{i_k}). Among the integer-valued solutions of the system, find the least and positive integer for p_{i_k}, which is what we want.

At last we shall determine a maximal subset of 1-cycles (in Type (iii)) which are linear independent with respect to homology. Take first one 1-cycle, S_1, of course it is linearly independent with respect to homology (l.i.w.r.t.h.), since it is not in Types (i) and (ii). Then proceed to examine next 1-cycle, S_2. Examine then whether S_1 and S_2 are l.i.w.r.t.h. Assume S_1 and S_2 are l.i.w.r.t.h. Then adjoin to them the next 1-cycle, say S_3, and verify whether S_1, S_2, and S_3 are l.i.w.r.t.h., and so forth, if the set $\{S_1, S_2, \cdots, S_\mu\}$ is l.i.w.r.t.h., then $\{S_i\}$ has only two types of sets, namely, (i) and (iii). It is obvious that $\nu = \mu$;

To prove whether a set $S_1, S_2, \ldots, S_J$ is l.i.w.r.t.h., consider the equation

$$\sum_{j=1}^{J} h_j \cdot S_j = \partial(\sum_{i=1}^{\alpha_2} g_i \cdot \sigma_i^2), \tag{9}$$

and write a corresponding system of linear algebraic equations (like in the preceding section) with unknowns g_i and h_j. Then the set $\{S_j\}$ is l.i.w.r.t.h.if and only if the system has no integer-valued nontrivial solution.

7. Writing out $H_1(K^2)$

By the preceding steps, we have determined parameters ν and p_i in the following expression

$$H_1(K^n) \cong \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{\nu} \oplus \mathbf{Z}_{p_1} \oplus \cdots \oplus \mathbf{Z}_{p_m}.$$

By Eq.(1), the method given as above is true.

8. Analyzing the time complexity of our method

In this section, we assume the triangulated n-complex K^n has p vertices and q edges. Then the number of basis $\{s_k\}$ of $Z_1(K^n)$ determined in Step 1 is $m := q - p + 1$. The time complexity in Step 1 is $O(m)$.

Then we assume the homology classes obtained in Step 2 is A_0, A_1, $A_2, \ldots, A_w$, A_{w+1}, $\ldots, A_\mu$, where $w \in \mathbf{Z}$. So the time complexity in Step 2 of our method is $O(m + m - 1 + \ldots + 2 + 1 + \mu) = O(\frac{m(m-1)}{2} + \mu)$.

From Step 3 to Step 4, we choose a representative element from A_0, A_1, $A_2, \ldots, A_w$, $A_{w+1}, \ldots, A_\mu$, respectively, and classfy them as follows

$$\underbrace{S_0}_{\text{Type (i)}}, \quad \underbrace{S_1, S_2, \ldots, S_w,}_{\text{Type (iii)}} \quad \underbrace{S_{w+1}, S_{w+2}, \ldots, S_\mu}_{\text{Type (ii)}}$$

the time complexity of our method is $O(w-1+w-2+\ldots+1+\mu-w) = O(\frac{w(w-1)}{2}+\mu-w)$.

Therefore, the complexity of our method is $O(m^2)$, where m is the number of the bases of $Z_1(K^n)$.

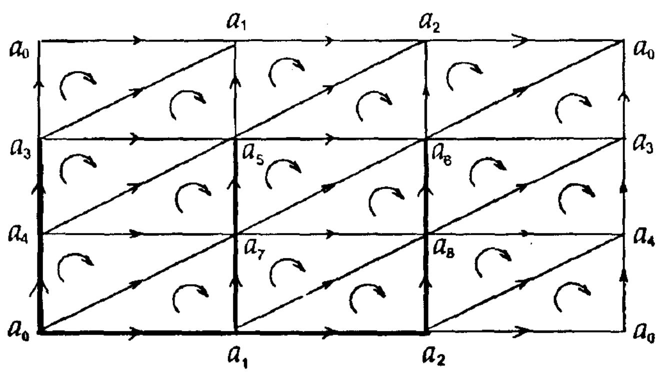

Figure 1: Triangulation of the torus

9. Applications

In this section, we shall use our method to compute the 1-dimensional homology groups of the torus S_1, the Klein bottle $\tilde{S}_2$, the 2-sphere S_0, the annulus A, the Möbius stripe M, the cylinder C, the project plane $\tilde{S}_1$, the nonorientable surface of genus three $\tilde{S}_3$.

The Torus is triangulated in Figure 1, the orientation is also indicated by the arrows. Let the heavy lines determine a spanning tree of S_1.

Since $m = p - q + 1 = 27 - 9 + 1 = 19$, list its basic cycles as follows:

$s_1 = a_0a_7a_1a_0$; $s_2 = a_0a_4a_7a_1a_0$; $s_3 = a_0a_4a_5a_7a_1a_0$;
$s_4 = a_0a_4a_3a_5a_7a_1a_0$; $s_5 = a_1a_8a_2a_1$; $s_6 = a_1a_7a_8a_2a_1$;
$s_7 = a_1a_7a_6a_8a_2a_1$; $s_8 = a_1a_7a_5a_6a_8a_2a_1$; $s_9 = a_6a_0a_1a_2a_8a_6$;
$s_{10} = a_6a_2a_8a_6$; $s_{11} = a_5a_2a_1a_7a_5$; $s_{12} = a_1a_7a_5a_1$;
$s_{13} = a_0a_4a_3a_1a_0$; $s_{14} = a_0a_4a_3a_0$; $s_{15} = a_6a_3a_4a_0a_1a_2a_8a_6$;
$s_{16} = a_8a_3a_4a_0a_1a_2a_8$; $s_{17} = a_8a_4a_0a_1a_2a_8$; $s_{18} = a_0a_1a_2a_0$;
$s_{19} = a_4a_0a_1a_2a_4$.

It is obvious that $s_i \sim 0, i \in \{1,2,3,4,5,6,7,8\}$, since each of them bounds a face.

$$\begin{aligned}
s_{10} =&\langle a_6a_2\rangle + \langle a_2a_8\rangle + \langle a_8a_6\rangle,\\
s_{11} =&\langle a_5a_2\rangle - \langle a_1a_2\rangle + \langle a_1a_7\rangle + \langle a_7a_5\rangle,\\
s_{12} =&\langle a_1a_7\rangle + \langle a_7a_5\rangle + \langle a_5a_1\rangle,\\
s_{13} =&\langle a_3a_1\rangle - \langle a_0a_1\rangle + \langle a_0a_4\rangle + \langle a_4a_3\rangle,\\
s_{14} =&\langle a_0a_4\rangle + \langle a_4a_3\rangle + \langle a_3a_0\rangle,\\
s_{15} =&\langle a_6a_3\rangle - \langle a_4a_3\rangle - \langle a_0a_4\rangle + \langle a_0a_1\rangle + \langle a_1a_2\rangle + \langle a_2a_8\rangle + \langle a_8a_6\rangle,\\
s_{16} =&\langle a_8a_3\rangle - \langle a_4a_3\rangle - \langle a_0a_4\rangle + \langle a_0a_1\rangle + \langle a_1a_2\rangle + \langle a_2a_8\rangle,\\
s_{17} =&\langle a_8a_4\rangle - \langle a_0a_4\rangle + \langle a_0a_1\rangle + \langle a_1a_2\rangle + \langle a_2a_8\rangle,\\
s_{18} =&\langle a_0a_1\rangle + \langle a_1a_2\rangle + \langle a_2a_0\rangle,\\
s_{19} =&\langle a_2a_4\rangle - \langle a_0a_4\rangle + \langle a_0a_1\rangle + \langle a_1a_2\rangle.
\end{aligned}$$

$$\begin{aligned}
s_{10} - s_{14} =&\langle a_6a_2\rangle + \langle a_2a_8\rangle + \langle a_8a_6\rangle - \langle a_0a_4\rangle - \langle a_4a_3\rangle - \langle a_3a_0\rangle\\
=&\partial(\langle a_6a_2a_0\rangle + \langle a_6a_0a_3\rangle + \langle a_8a_6a_3\rangle + \langle a_8a_3a_4\rangle + \langle a_2a_8a_4\rangle + \langle a_6a_0a_3\rangle + \langle a_2a_4a_0\rangle),
\end{aligned}$$

hence $s_{10} \sim s_{14}$.

$$\begin{aligned}
s_{11} - s_{14} =&\langle a_5a_2\rangle - \langle a_1a_2\rangle + \langle a_1a_7\rangle - \langle a_7a_5\rangle - \langle a_0a_4\rangle - \langle a_4a_3\rangle - \langle a_3a_0\rangle\\
=&\partial(\langle a_5a_2a_6\rangle + \langle a_2a_0a_6\rangle + \langle a_7a_5a_6\rangle + \langle a_7a_6a_8\rangle + \langle a_8a_6a_3\rangle + \langle a_8a_3a_4\rangle + \langle a_1a_7a_8\rangle +\\
&\langle a_1a_8a_2\rangle + \langle a_2a_8a_4\rangle + \langle a_6a_0a_3\rangle + \langle a_2a_4a_0\rangle),
\end{aligned}$$

therefore $s_{11} \sim s_{14}$.

$$\begin{aligned}
s_{12} - s_{14} =&\langle a_1a_7\rangle + \langle a_7a_5\rangle + \langle a_5a_1\rangle - \langle a_0a_4\rangle - \langle a_4a_3\rangle - \langle a_3a_0\rangle\\
=&\partial(\langle a_1a_2a_5\rangle + \langle a_5a_2a_6\rangle + \langle a_2a_0a_6\rangle + \langle a_6a_0a_3\rangle + \langle a_7a_5a_6\rangle + \langle a_7a_6a_8\rangle + \langle a_8a_6a_3\rangle +\\
&\langle a_8a_3a_4\rangle + \langle a_1a_7a_8\rangle + \langle a_1a_8a_2\rangle + \langle a_2a_8a_4\rangle + \langle a_2a_4a_0\rangle),
\end{aligned}$$

so $s_{12} \sim s_{14}$.

$$\begin{aligned}
s_{15} - s_{18} =&\langle a_6a_3\rangle - \langle a_4a_3\rangle - \langle a_0a_4\rangle - \langle a_2a_0\rangle + \langle a_2a_8\rangle + \langle a_8a_6\rangle\\
=&\partial(\langle a_8a_6a_3\rangle + \langle a_8a_3a_4\rangle + \langle a_2a_8a_4\rangle + \langle a_2a_4a_0\rangle),
\end{aligned}$$

so we obtain $s_{15} \sim s_{18}$.

$$s_{16} - s_{18} = \langle a_8a_3\rangle - \langle a_4a_3\rangle - \langle a_0a_4\rangle + \langle a_2a_8\rangle - \langle a_2a_{10} = \partial(\langle a_8a_3a_4\rangle + \langle a_2a_8a_4\rangle - \langle a_2a_0a_4\rangle),$$

hence $s_{16} \sim s_{18}$.

$$s_{17} - s_{18} = \langle a_8 a_4 \rangle - \langle a_0 a_4 \rangle + \langle a_2 a_8 \rangle - \langle a_2 a_0 \rangle = \partial(\langle a_2 a_8 a_4 \rangle + \langle a_2 a_4 a_0 \rangle),$$

so that $s_{17} \sim s_{18}$.

$$s_{19} - s_{18} = -\langle a_2 a_0 \rangle - \langle a_0 a_4 \rangle + \langle a_2 a_4 \rangle = \partial(\langle a_2 a_0 a_4 \rangle),$$

hence $s_{19} \sim s_{18}$.

Next, we shall prove $s_{18} \not\sim s_{14}$.

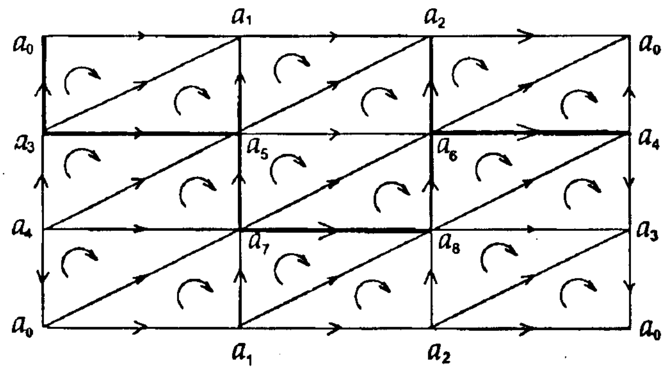

Figure 2: Triangulation of the Klein bottle

Otherwise, assume $s_{18} \sim s_{14}$, that is to say there exists c_2 such that $\partial(c_2) = s_{18} - s_{14}$, namely $\langle a_0 a_1 \rangle + \langle a_1 a_2 \rangle + \langle a_2 a_0 \rangle - \langle a_0 a_4 \rangle - \langle a_4 a_3 \rangle - \langle a_3 a_0 \rangle = \partial(\sum_{i=1}^{18} g_i \cdot \sigma_i^2)$. Consider any 1-simplex which is not in s_{18} and s_{14}, say $\langle a_3 a_1 \rangle$, $\partial(g_1 \cdot \langle a_3 a_0 a_1 \rangle + g_2 \cdot \langle a_1 a_5 a_3 \rangle + \cdots) = -g_1 \cdot \langle a_3 a_1 \rangle + g_2 \cdot \langle a_3 a_1 \rangle + \underbrace{\cdots}_{no \langle a_3 a_1 \rangle}$, this implies $g_1 - g_2 = 0$, namely $g_1 = g_2$. So we obtain $g_i = g \in s$ for $\forall i \in \{1, 2, \cdots 18\}$, then $c_2 = \sum_{i=1}^{18} g \cdot \sigma_i^2$, whence $s_{18} - s_{14} = \partial(c_2)$. Fix on some 1-simplex which is in s_{14} or s_{18}, say $\langle a_3 a_4 \rangle$, $\partial(g \cdot \langle a_3 a_5 a_4 \rangle + g \cdot \langle a_4 a_3 a_8 \rangle + \cdots) = -g \cdot \langle a_4 a_3 \rangle + g \cdot \langle a_4 a_3 \rangle + \cdots = 0$. This implies $\partial(c_2) = 0$, namely $s_{18} = s_{14}$, which is a contradiction to the assumption. The similar arguments yield: $k \cdot s_{18} \not\sim 0$, $k \cdot s_{14} \not\sim 0$. But $s_9 - s_{14} - s_{18} = \langle a_6 a_0 \rangle + \langle a_2 a_8 \rangle + \langle a_8 a_6 \rangle - \langle a_0 a_4 \rangle - \langle a_4 a_3 \rangle - \langle a_3 a_0 \rangle - \langle a_2 a_0 \rangle = \partial(\langle a_6 a_0 a_3 \rangle + \langle a_8 a_6 a_3 \rangle + \langle a_8 a_3 a_4 \rangle + \langle a_2 a_8 a_4 \rangle + \langle a_2 a_4 a_0 \rangle)$, this implies $s_{14} - s_{18} + s_9 \sim 0$.

From above, we partition the basis into four homology classes, they are $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, $\{s_9\}$, $\{s_{10}, s_{11}, s_{12}, s_{14}\}$, $\{s_{15}, s_{16}, s_{17}, s_{18}, s_{19}\}$, choose s_1, s_9, s_{14}, s_{18} as one representative of them, respectively, since $s_1 \sim 0$, $s_9 - s_{14} + s_{18} \sim 0$, and s_{14}, s_{18} are l.i.w.r.t.h., we obtain $\nu'' = 2$, and hence $H_1(S_1) \cong \mathbf{Z} \oplus \mathbf{Z}$.

$\tilde{S}_2$ is triangulated as shown in Figure 2, the orientation is indicated by arrows, the heavy lines determine a spanning tree of the Klein bottle.

List the basic cycles as following:

$s_1 = a_0 a_1 a_5 a_3 a_0$;	$s_2 = a_3 a_1 a_5 a_3$;	$s_3 = a_1 a_2 a_6 a_8 a_7 a_5 a_1$;
$s_4 = a_5 a_2 a_6 a_8 a_7 a_5$;	$s_5 = a_5 a_6 a_8 a_7 a_5$;	$s_6 = a_7 a_6 a_8 a_7$;
$s_7 = a_6 a_4 a_8 a_6$;	$s_8 = a_2 a_0 a_3 a_5 a_7 a_8 a_6 a_2$;	$s_9 = a_6 a_0 a_3 a_5 a_7 a_8 a_6$;
$s_{10} = a_4 a_0 a_3 a_5 a_7 a_8 a_6 a_4$;	$s_{11} = a_4 a_3 a_5 a_7 a_8 a_6 a_4$;	$s_{12} = a_3 a_8 a_7 a_5 a_3$;
$s_{13} = a_2 a_8 a_6 a_2$;	$s_{14} = a_2 a_3 a_5 a_7 a_8 a_6 a_2$;	$s_{15} = a_1 a_8 a_7 a_5 a_1$;

$s_{16} = a_1a_7a_5a_1$; $\quad s_{17} = a_0a_7a_5a_3a_0$; $\quad s_{18} = a_4a_7a_8a_6a_4$;
$s_{19} = a_4a_5a_7a_8a_6a_4$.

We can prove: $s_i \sim 0, i \in \{1, \cdots, 7\}$, since each basic cycle bounds a face. Moreover, $s_8 \sim s_{10}, s_9 \sim s_{10}, s_{14} \sim s_{10}$; $s_{12} \sim s_{11}, s_{18} \sim s_{11}, s_{19} \sim s_{11}$; $s_{13} \sim s_{17}, s_{15} \sim s_{17}, s_{16} \sim s_{17}$. In fact,

$$
\begin{aligned}
s_8 =& \langle a_2a_0\rangle - \langle a_3a_0\rangle + \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_2\rangle,\\
s_9 =& \langle a_6a_0\rangle - \langle a_3a_0\rangle + \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle,\\
s_{10} =& \langle a_4a_0\rangle - \langle a_3a_0\rangle + \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_4\rangle,\\
s_{11} =& \langle a_4a_3\rangle + \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_4\rangle,\\
s_{12} =& \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_3\rangle,\\
s_{14} =& \langle a_2a_3\rangle + \langle a_3a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_2\rangle,\\
s_{18} =& \langle a_4a_7\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_4\rangle,\\
s_{19} =& \langle a_4a_5\rangle - \langle a_7a_5\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_4\rangle.\\
s_8 - s_{10} =& \langle a_2a_0\rangle + \langle a_6a_2\rangle - \langle a_4a_0\rangle - \langle a_6a_4\rangle = \partial(\langle a_6a_0a_4\rangle + \langle a_6a_2a_0\rangle),
\end{aligned}
$$

so we obtain $s_8 \sim s_{10}$.

$$s_9 - s_{10} = \langle a_6a_0\rangle - \langle a_4a_0\rangle - \langle a_6a_4\rangle = \partial(\langle a_6a_0a_4\rangle),$$

hence $s_9 \sim s_{10}$.

$$
\begin{aligned}
s_{14} - s_{10} =& \langle a_2a_3\rangle + \langle a_6a_2\rangle - \langle a_4a_0\rangle + \langle a_3a_0\rangle - \langle a_3a_0\rangle - \langle a_6a_4\rangle\\
=& \partial(\langle a_2a_3a_0\rangle + \langle a_6a_2a_0\rangle + \langle a_6a_0a_4\rangle),
\end{aligned}
$$

so we get $s_{14} \sim s_{10}$.

$$s_{12} - s_{11} = \langle a_8a_3\rangle - \langle a_4a_3\rangle - \langle a_8a_6\rangle - \langle a_6a_4\rangle = \partial(-\langle a_6a_4a_8\rangle - \langle a_8a_4a_3\rangle),$$

hence $s_{12} \sim s_{10}$.

$$s_{18} - s_{11} = \langle a_4a_7\rangle - \langle a_4a_3\rangle - \langle a_3a_5\rangle + \langle a_7a_5\rangle = \partial(-\langle a_3a_5a_4\rangle - \langle a_4a_5a_7\rangle),$$

namely $s_{18} \sim s_{11}$.

$$s_{19} - s_{11} = \langle a_4a_5\rangle - \langle a_4a_3\rangle - \langle a_3a_5\rangle = \partial(-\langle a_3a_5a_4\rangle),$$

so we get $s_{19} \sim s_{11}$.

$$2 \cdot s_{11} - 2 \cdot s_{10} = 2 \cdot (\langle a_4a_3\rangle + \langle a_3a_0\rangle - \langle a_4a_0\rangle) = \partial(\sum_{i=1}^{18} \sigma_i^2),$$

this implies $2 \cdot s_{10} - 2 \cdot s_{11} \sim 0$ (homologous to zero).

$$\begin{aligned}
s_{13} =& \langle a_2a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_2\rangle,\\
s_{15} =& \langle a_1a_8\rangle - \langle a_7a_8\rangle + \langle a_7a_5\rangle + \langle a_5a_1\rangle,\\
s_{16} =& \langle a_1a_7\rangle + \langle a_7a_5\rangle + \langle a_5a_1\rangle,\\
s_{17} =& \langle a_0a_7\rangle + \langle a_7a_5\rangle - \langle a_3a_5\rangle + \langle a_3a_0\rangle.\\
s_{13} - s_{17} =& \langle a_2a_6\rangle + \langle a_8a_6\rangle + \langle a_6a_2\rangle - \langle a_0a_7\rangle - \langle a_7a_5\rangle - \langle a_3a_5\rangle - \langle a_3a_0\rangle\\
=& \partial(-(\langle a_0a_7a_1\rangle + \langle a_1a_7a_8\rangle + \langle a_1a_8a_2\rangle + \langle a_7a_6a_8\rangle + \langle a_7a_5a_6\rangle +\\
& \langle a_5a_2a_6\rangle + \langle a_1a_2a_5\rangle + \langle a_3a_1a_5\rangle + \langle a_3a_0a_1\rangle)),
\end{aligned}$$

so we get $s_{13} \sim s_{17}$.

$$\begin{aligned}
s_{15} - s_{17} =& \langle a_1a_8\rangle - \langle a_7a_8\rangle + \langle a_5a_1\rangle - \langle a_0a_7\rangle + \langle a_3a_5\rangle - \langle a_3a_0\rangle\\
=& \partial(-(\langle a_3a_0a_1\rangle + \langle a_3a_1a_5\rangle + \langle a_0a_7a_1\rangle + \langle a_1a_7a_8\rangle)),
\end{aligned}$$

hence $s_{15} \sim s_{17}$.

$$\begin{aligned}
s_{16} - s_{17} =& - \langle a_0a_7\rangle + \langle a_3a_5\rangle + \langle a_3a_0\rangle + \langle a_1a_7\rangle + \langle a_5a_1\rangle\\
=& \partial(-\langle a_0a_1a_3\rangle - \langle a_3a_1a_5\rangle + \langle a_0a_1a_7\rangle),
\end{aligned}$$

so that $s_{16} \sim s_{17}$.

Then we shall prove, $s_{10} \not\sim \pm s_{17}$. In fact, assume $s_{10} \sim -s_{17}$, then there exists c_2 such that $s_{10} - (-s_{17}) = \partial(c_2)$, namely $\langle a_4a_0\rangle + \langle a_7a_8\rangle + \langle a_8a_6\rangle + \langle a_6a_4\rangle - \langle a_0a_7\rangle = \partial(\sum_{i=1}^{18} g_i \cdot \sigma_i^2)$. Fix 1-simplex which is not in s_{10} and $-s_{17}$, say $\langle a_3a_5\rangle$, then $\partial(g_1 \cdot \langle a_3a_1a_5\rangle + g_2 \cdot \langle a_3a_5a_4\rangle + \cdots) = -g_1 \cdot \langle a_3a_5\rangle + g_2 \cdot \langle a_3a_5\rangle + \cdots + \underbrace{\cdots}_{no\langle a_3a_5\rangle}$. We easily get $g_1 - g_2 = 0$, this means all the coefficients of 2-simplex have a common value, say g, then $s_{10} - (-s_{17}) = \partial(\sum_{i=1}^{18} g \cdot \sigma_i^2)$. But if we fix some 1-simplex which is in s_{10} or s_{17}, say $\langle a_7a_8\rangle$, $\partial(g \cdot \langle a_7a_6a_8\rangle + g \cdot \langle a_7a_8a_1\rangle + \cdots) = -g \cdot \langle a_7a_8\rangle + g \cdot \langle a_7a_8\rangle + \cdots$, which implies $s_{10} - (-s_{17}) = 0$, namely $s_{10} = -s_{17}$, a contradiction arises. So $s_{10} \not\sim -s_{17}$, the similar argument yields: $s_{10} \not\sim s_{17}$.

In the following, we shall prove : $s_{17} \not\sim 0; s_{10} \not\sim 0; g \cdot s_{10} \not\sim 0, g \in \mathbf{Z}; 2 \cdot s_{17} \sim 0$. We assume $s_{17} \sim 0$, that's to say, there exists c_2, such that $\partial(c_2) = s_{17}$, namely

$$\langle a_0a_7\rangle + \langle a_7a_5\rangle - \langle a_3a_5\rangle + \langle a_3a_0\rangle = \partial(\sum_{i=1}^{18} g_i \cdot \sigma_i^2). \qquad (10)$$

Let's consider any 1-simplex which is not in s_{17}, for example $\langle a_6a_4\rangle$, $\partial(g_1 \cdot \langle a_4a_8a_6\rangle + g_2\langle a_6a_0a_4\rangle + \cdots) = g_1 \cdot \langle a_6a_4\rangle - g_2 \cdot \langle a_6a_4\rangle + \underbrace{\cdots}_{no\langle a_6a_4\rangle}$. From the foregoing procedure, we get: $g_1 - g_2 = 0$, namely $g_1 = g_2$, which implies $g_i = g$, $i \in \{1, 2, \cdots, 18\}$. So we get $c_2 = \sum_{i=1}^{18} g \cdot \sigma_i^2$ satisfying $\partial(c_2) = s_{17}$. But if we fix another 1-simplex which is in s_{17}, say $\langle a_0a_7\rangle$, then $\partial(g \cdot \langle a_0a_7a_1\rangle + g \cdot \langle a_0a_4a_7\rangle + \cdots) = g \cdot \langle a_0a_7\rangle - g \cdot \langle a_0a_7\rangle + \cdots = 0$, the left-hand side of Eq.(10) with no term $\langle a_0a_7\rangle$. But this is impossible , hence $s_{17} \not\sim 0$. Similarly, we get: $s_{10} \not\sim 0, g \cdot s_{10} \not\sim 0, g \in \mathbf{Z}$, namely s_{10} is l.i.w.r.t.h.

Finally, we shall prove: $2\cdot s_{17}\sim 0$. Since $2\cdot s_{17}=2\cdot(\langle a_0a_7\rangle+\langle a_7a_5\rangle-\langle a_5a_3\rangle+\langle a_3a_0\rangle)=\partial(\langle a_0a_1a_3\rangle+\langle a_3a_1a_5\rangle+\langle a_1a_2a_5\rangle+\langle a_5a_2a_6\rangle+\langle a_2a_0a_6\rangle+\langle a_6a_0a_4\rangle+\langle a_7a_5a_6\rangle+\langle a_7a_6a_8\rangle+\langle a_8a_6a_4\rangle+\langle a_8a_4a_3\rangle+\langle a_0a_7a_1\rangle+\langle a_1a_7a_8\rangle+\langle a_1a_8a_2\rangle+\langle a_2a_8a_3\rangle+\langle a_2a_3a_0\rangle-\langle a_3a_5a_4\rangle-\langle a_4a_5a_7\rangle-\langle a_4a_7a_0\rangle)$.

By the preceding course, we partition the basis into four homology classes, they are $\{s_1,s_2,s_3,s_4,s_5,s_6,s_7\}$, $\{s_{11},s_{12},s_{18},s_{19}\}$, $\{s_8,s_9,s_{10},s_{14}\}$, $\{s_{13},s_{15},s_{16},s_{17}\}$, choose s_1,s_{10},s_{11}, s_{17} as one representative of them, respectively. Since $s_1\sim 0$, $s_{10}-s_{11}+s_{17}=\partial(-\langle a_3a_5a_4\rangle-\langle a_4a_5a_7\rangle-\langle a_4a_7a_0\rangle)$, but $2\cdot s_{17}\sim 0$, $2\cdot(s_{10}-s_{11})\sim 0$, s_{10} is l.i.w.r.t.h. so $\nu=1,p_1=2$. Hence $H_1(\tilde{S}_2)\cong \mathbf{Z}\oplus\mathbf{Z}_2$.

With the same method as we have given, we can also get:

$$H_1(S_0)=\{0\},H_1(A)\cong\mathbf{Z},H_1(M)\cong\mathbf{Z},H_1(C)\cong\mathbf{Z},H_1(P)\cong\mathbf{Z}_2,H_1(\tilde{S}_3)\cong\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}_2.$$

10. Open problem

The preceding investigation of the method for computing $H_1(K^2)$ is quite detailed. Here we present an open problem: How can we generalize the method to compute the 2-dimensional homology group of a given n-complex?

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References:

[1] BONDY J, MURTY U S R. *Graphy Theory with Applications* [M]. MacMillan, London, 1976.

[2] CROOM F H. *Basic Concepts of Algebraic Topology* [M]. Springer-Verlag, New York, 1978.

[3] HARARY F. *Graphy Theory* [M]. Addison - Wesley, Reading, Mass, 1969.

[4] VILLARROEL-FLORES R, WEBB P. *Some split exact sequences in the cohomology of groups* [J]. Topology, 2002, 41(3): 483–494.

[5] WEBB P J. *A local method in group cohomology* [J]. Comment. Math. Helv., 1987, 62(1): 135–167.

一种计算给定 2- 复型 1- 维同调群的新方法

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摘 要: 文献 [3] 在有限域 $\mathbf{Z}_2$ 上描述了图 G 的 "圈空间". 这里我们将此理论推广到一般环 $\mathbf{Z}$ 上用以计算给定 2- 复型的一维同调群, 其中我们采用的方法是代数与图论相结合的方法.

关键词: 1- 链; 1- 圈; 复形; 同调群