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Generalized Steiner Triple Systems with Group Size Ten *

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Abstract: Generalized Steiner triple systems, $\mathrm{GS}(2,3,n,g)$ are equivalent to $(g+1)$-ary maximum constant weight codes $(n,3,3)$s. In this paper, it is proved that the necessary conditions for the existence of a $\mathrm{GS}(2,3,n,10)$, namely, $n \equiv 0,1 \pmod 3$ and $n \geq 12$, are also sufficient.

Key words: generalized Steiner triple system; constant weight codes; holey generalized Steiner triple system; singular indirect product.

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1. Introduction

A $(g+1)$-ary *constant weight code* (n,w,d) is a code $C \subseteq (Z_{g+1})^n$ of length n and minimum distance d, such that every $c \in C$ has Hamming weight w. To construct a constant weight code (n,w,d) with $w=3$, a *group divisible design* (GDD) will be used. A K−GDD is an ordered triple $(\mathcal{V},\mathcal{G},\mathcal{B})$ where $\mathcal{V}$ is a set of n elements, $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called *groups* which partition $\mathcal{V}$, and $\mathcal{B}$ is a set of some subsets of $\mathcal{V}$ called *blocks*, such that each block intersects each group in at most one element and that each pair of elements from distinct groups occurs together in exactly one block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. The group type is the multiset $\{|G| : G \in \mathcal{G}\}$. A k-GDD(g^n) denotes a K-GDD with n groups of size g and $K=\{k\}$. In a 3-GDD(g^n), let $\mathcal{V}=(Z_{g+1}\setminus\{0\})\times(Z_{n+1}\setminus\{0\})$ with n groups $G_i \in \mathcal{G}$, $G_i=(Z_{g+1}\setminus\{0\})\times\{i\}$, $1 \leq i \leq n$ and blocks $\{(a,i),(b,j),(c,k)\} \in \mathcal{B}$. One can construct a constant weight c

ode $(n,3,d)$ as stated in [1], [2]. From each block we form a codeword of length n by putting an a, b and c in positions i, j and k respectively and zeros elsewhere. This gives a constant weight code over Z_{g+1} with minimum distance 2 or 3. If the minimum distance is 3, then the code is a $(g+1)$-ary *maximum constant weight code* (MCWC) $(n,3,3)$ and

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the 3-GDD(g^n) is called *generalized Steiner triple system*, denoted by GS$(2,3,n,g)$. It is easy to see that a 3-GDD(g^n) is a GS$(2,3,n,g)$ iff any two intersecting blocks intersect at most two common groups of the GDD. The following result is known.

Lemma 1.1[1,2] *If there exists a* GS$(2,3,n,g)$, *then*

(1) $(n-1)g \equiv 0 \pmod 2$;

(2) $n(n-1)g^2 \equiv 0 \pmod 6$;

(3) $n \geq g+2$.

The necessary conditions are shown to be sufficient for $g = 2,3$ with one exception by Etzion[1], for $g = 4,9$ by Phelps and Yin[1,2], for $g = 5,6$ by Chen, Ge and Zhu[4,5], for $g = 7,8$ by Wu, Ge and Zhu[6].

Lemma 1.2 *The necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for* g = 2, 3, 4, 5, 6, 7, 8 *and* 9 *with one exception of* $(g,n) = (2,6)$.

Blake-Wilson and Phelps[7] proved that the necessary conditions for the existence of a GS$(2,3,n,g)$ are also asymptotically sufficient for any g. As used in [6], for $g \geq 7$, let $T_g = \{n$: there exists a GS$(2,3,n,g)\}$, $B_g = \{n$: n satisfying the necessary conditions listed in Lemma 1.1 $\}$, $M_g = \{n\colon n \in B_g, n \leq 9g+158\}$. We have the following.

Lemma 1.3[6] *For any* $g \geq 7$, *if* $M_g \subset T_g$, *then* $B_g = T_g$. *That is the necessary conditions for the existence of a* GS$(2,3,n,g)$ *are also sufficient.*

In this paper, the following result is obtained.

Theorem 1.4 *There exists a* GS$(2,3,n,10)$ *if and only if* $n \equiv 0,1 \pmod 3$ *and* $n \geq 12$.

Combining Lemma 1.2 and Theorem 1.4, it is known that the existence of a GS$(2,3,n,g)$ is completely determined for any $g \leq 10$. For general background on designs, see [8].

2. Preliminaries

In product constructions, we will need the concept of both *holey generalized Steiner triple systems* and *disjoint incomplete Latin squares.*

A *holey group divisible design*, $K-HGDD$, is a fourtuple $(\mathcal{V},\mathcal{G},\mathcal{H},\mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. A k-HGDD($g^{(n,u)}$) denotes a K-HGDD with n groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K = \{k\}$. A *holey generalized Steiner triple system*, HGS$(2,3,(n,u),g)$, is a 3-HGDD($g^{(n,u)}$) with the property that any two intersecting blocks intersect at most two common groups.

It is easy to see that if $u = 0$ or $u = 1$, then a HGS$(2,3,(n+u,u),g)$ is just a GS$(2,3,n,g)$ or a GS$(2,3,n+1,g)$ respectively.

A *Latin square* of side n, LS(n), is an $n \times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. An *incomplete Latin square*, ILS$(n+a,a)$, denotes a LS$(n+a)$ "missing" a sub LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $(i,j,s) \in$ ILS$(n+a,a)$ if the entry in the cell (i,j) is s. Let A_1, A_2 be two ILS$(n+a,a)$s on the same symbol set. If $(i,j,s_1) \neq (i,j,s_2)$ for any $(i,j,s_1) \in A_1$, $(i,j,s_2) \in A_2$, then we say that A_1 and A_2 are *disjoint*. We use r DILS$(n+a,a)$ to denote r pairwise disjoint ILS$(n+a,a)$s, and r DLS(n) to denote r pairwise disjoint LS(n)s.

The following singular indirect product construction was first stated in [4].

Lemma 2.1 (SIP) *Let m,n,t,u and a be integers such that $0 \leq a \leq u < n$. Suppose the following designs exist: (1) tDILS$(n+a,a)$; (2) a $3-$GDD(g^m) with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3; (3) a HGS$(2,3,(n+u,u),g)$. Then there exists a HGS$(2,3,(c,d),g)$, where $c = m(n+a)+u-a, d = ma+u-a$. Further, if there exists (4) a GS$(2,3,ma+u-a,g)$, then there exists a GS$(2,3,m(n+a)+u-a,g)$.*

To use SIP construction, we need the following known result on t DILS$(n+a,a)$.

Lemma 2.2[9] *There exist nDILS$(n+a,a)$ for any positive integer n and for any integer a, $0 \leq a \leq n$ except for $(n,a) = (2,1)$, $(6,5)$.*

From Lemma 2.2, t DLS(n) exist when $t \leq n$. So, take $u \in \{0,1\}$, $a = 0$ in Lemma 2.1, we have the following.

Lemma 2.3 *Let m,n,t, and u be integers such that $u \in \{0,1\}$. Suppose $t \leq n$ and the following designs exist: (1) a $3-$GDD(g^m) with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3; (2) a GS$(2,3,n+u,g)$. Then there exist both an HGS$(2,3,(mn+u,u),g)$ and a GS$(2,3,mn+u,g)$.*

The following lemma is similar to Lemma 5.8 in [6], so we omit the proof.

Lemma 2.4 *If there exists a GS(2,3,n,10), then there exist a GS(2, 3, mn,10) and a GS(2, 3, $m(n-1)+1$, 10), where $m = 3,4,6$ and 7.*

3. Proof of Theorem 1.4

For Lemma 1.1, the necessary conditions for the existence of a GS$(2,3,n,10)$ become $n \equiv 0,1 \pmod 3$ and $n \geq 12$. It is known that there exists a GS$(2,3,q+1,q-1)$ for any prime power q in [1, Section 4]. Take $q = 11$, we get a GS$(2,3,12,10)$.

Lemma 3.1 *There exists a GS$(2,3,n,10)$ for any $n \in F_1$, where F_1={12,13, 16, 21, 24, 25, 28, 33, 40}.*

Proof For $n = 12$, as stated above, there exists a GS$(2,3,12,10)$. For each $n \in F_1 \setminus \{12\}$, with the aid of a computer, we have found a set of base blocks of a GS$(2,3,n,10)$. The corresponding base blocks are listed in Appendix A (In order to save space, we omit Appendix A, the interested reader may contact the authors for a copy). □

Lemma 3.2 *There exists a GS$(2,3,n,10)$ for any $n \in F_2$, where F_2={18, 22, 30, 42, 58}.*

Proof For each $n \in F_2$, with the aid of a computer, we have found a set of generalized base blocks of a GS$(2,3,n,10)$. The corresponding base blocks are listed in Appendix B (In order to save space, we omit Appendix B, the interested reader may contact the authors for a copy). □

Lemma 3.3 *There exists a* GS$(2,3,n,10)$ *for any* $n \in F_3$, *where* $F_3=\{15, 19, 27, 31, 51\}$.

Proof For each $n \in F_3$, with the aid of a computer, we have found a set of generalized base blocks of a GS$(2,3,n,10)$. The corresponding base blocks are listed in Appendix C. (In order to save space, we omit Appendix C, the interested reader may contact the authors for a copy). □

Lemma 3.4 *There exists a* GS$(2,3,v,10)$ *for any* $v \in F_4$, *where* $F_4 = \{v : v \equiv 0,1 \ (mod\ 3), 12 \le v \le 82\}$.

Proof For $v \in F_1 \cup F_2 \cup F_3$, the conclusion comes from Lemmas 3.1-3.3. For the remaining values v, we can write $v = mn$ or $v = m(n-1)+1$ for some $m \in \{3,4,6\}$ and $n \in F_1 \cup F_2 \cup F_3$. By Lemmas 3.1–3.3 and Lemma 2.4, there exists a GS$(2,3,v,10)$. Here, we list the triples (v,m,n) in Table 3.1. □

v	m	n	v	m	n	v	m	n
$34=3\cdot 11+1$	3	12	$36=3\cdot 12$	3	12	$37=3\cdot 12+1$	3	13
$39=3\cdot 13$	3	13	$43=3\cdot 14+1$	3	15	$45=3\cdot 15$	3	15
$46=3\cdot 15+1$	3	16	$48=3\cdot 16$	3	16	$49=4\cdot 12+1$	4	13
$52=3\cdot 17+1$	3	18	$54=3\cdot 18$	3	18	$55=3\cdot 18+1$	3	19
$57=3\cdot 19$	3	19	$60=4\cdot 15$	4	15	$61=4\cdot 15+1$	4	16
$63=3\cdot 21$	3	21	$64=3\cdot 21+1$	3	22	$66=3\cdot 22$	3	22
$67=6\cdot 11+1$	6	12	$69=4\cdot 17+1$	4	18	$70=3\cdot 23+1$	3	24
$72=3\cdot 24$	3	24	$73=3\cdot 24+1$	3	25	$75=3\cdot 25$	3	25
$76=4\cdot 19$	4	19	$78=6\cdot 13$	6	13	$79=3\cdot 26+1$	3	27
$81=3\cdot 27$	3	27	$82=3\cdot 27+1$	3	28			

Table 3.1 triples (v,m,n) for $v \in F_4 \setminus (F_1 \cup F_2 \cup F_3)$

Lemma 3.5 *There exists a* GS$(2,3,v,10)$ *for any* $v \in F_5$, *where* $F_5 = \{v\colon v \equiv 0,1,3,7 \ (mod\ 9),\ 12 \le v \le 246\}$.

Proof For $v \equiv 0,1,3 \pmod 9$, write $v = 9t+k$, where $k = 0,1,3$. If $t \le 3$, the result follows from Lemma 3.4. Otherwise, $t \ge 4$. Let $n = 3t$, then $v = 3n, 3n+1$ or $3(n+1)$. Since $v \le 246$, we have $4 \le t \le 27$, hence $n \le 81, n+1 \le 82$. Notice that $n \in B_{10}$ and $n+1 \in B_{10}$, by Lemma 2.4 and Lemma 3.4, there exists a GS$(2,3,v,10)$.

For $v \equiv 7 \pmod 9$, write $v = 9t+7$. If $t \le 2$, the result follows from Lemma 3.4. Otherwise, $t \ge 3$. Let $n = 3t+3$, then $v = 3(n-1)+1$. Since $v \le 246$, we have $t \le 26$, hence $n \le 81$. Notice that $n \in B_{10}$, by Lemma 2.8 and Lemma 3.4, there exists a GS$(2,3,v,10)$. □

Lemma 3.6 *There exists a* GS$(2,3,v,10)$ *for any* $v \in F_6$, *where* $F_6 = \{v\colon v \equiv 4,6,13,24,31,33 \pmod{36}, 12 \le v \le 247\}$.

Proof Write $v = 36t+k, k = 4,6,13,24,31,33$. If $t = 0$ and $k \ge 24$ or $t \le 1$ and $k \le 13$, the result comes from Lemma 3.4. Otherwise, $t \ge 1$ for $k \ge 24$ or $t \ge 2$ for $k \le 13$. Notice $v \le 247$, we can write $v = mn$ or $v = mn+1$ for some $m \in \{4,6\}$ and $n \in B_{10}, n \le 60$. From Lemma 2.4 and Lemma 3.4, there exists a GS$(2,3,v,10)$. Here we list the fourtuples (k,v,m,n) in Table 3.2 . □

k	v	m	n	k	v	m	n
4	$v = 4 \cdot (9t+1)$	4	9t+1	6	$v = 6 \cdot (6t+1)$	6	6t+1
13	$v = 4 \cdot (9t+3)+1$	4	9t+4	24	$v = 6 \cdot (6t+4)$	6	6t+4
31	$v = 6 \cdot (6t+5)+1$	6	6t+6	33	$v = 4 \cdot (9t+8)+1$	4	9t+9

Table 3.2 fourtuples (k, v, m, n) for Lemma 3.6

Lemma 3.7 *There exists a* GS$(2,3,v,10)$ *for any* $v \in F_7$, *where* $F_7 = \{v\colon v \equiv 15, 22 \pmod{36}, 12 \le v \le 247\}$.

Proof For $v \equiv 15 \pmod{36}$, write $v = 36e + 15$. If $e \le 1$, then $v \le 51$, from Lemma 3.4, there exists a GS$(2,3,51,10)$. If $e = 2$, then $v = 87$. Since there exists a GS$(2,3,13,10)$ by Lemma 3.4, we get an HGS$(2,3,(37,13),10)$ by Lemma 2.4. We can apply Lemma 2.1 with $m = 3, n = 24, t = 10, u = 13, a = 1$ to obtain a GS$(2,3,87,10)$. The 10 DILS$(24+1,1)$ comes from Lemma 2.2, and the GS$(2,3,15,10)$ is from Lemma 3.4. For $e \ge 3$, take $u = 3e+3$ and $n = 6e+6$, then $3e-4 \ge 5$. Since $e \ge 3$ and $v \le 247$, we have $3 \le e \le 6$, hence $12 \le u \le 21$. From Lemma 3.4, there exists a GS$(2,3,u,10)$. So there exists an HGS$(2,3,(n+u,u),10)$ from Lemma 2.4. Apply Lemma 2.1 with $m = 4, n = 6e+6, t = 5, u = 3e+3, a = 3e-4$, we obtain a GS$(2,3,v,10)$. The 5 DILS$(n+a,a)$ comes from Lemma 2.2 since $3e-4 \ge t$. The GS$(2,3,ma+u-a,10)$ is from Lemma 3.4, since $27 \le ma+u-a = 12e-9 \le 63$.

For $v \equiv 22 \pmod{36}$, write $v = 36e+22$. For $e = 1$, the result follows from Lemma 3.4. For $e = 2$, $v = 94$, apply Lemma 2.1 with $m = 3, n = 24, t = 10, u = 12, a = 5$, we get a GS$(2,3,94,10)$. For $e \ge 3$. Just as we did in the case $v \equiv 15 \pmod{36}$, apply Lemma 2.1 with $m = 4, n = 6e+6, t = 5, u = 3e+4, a = 3e-2$, we obtain a GS$(2,3,v,10)$. □

Combining Lemma 3.6 and Lemma 3.7, we have the following.

Lemma 3.8 *There exists a* GS$(2,3,v,10)$ *for any* $v \in F_8$, *where* $F_8 = \{v\colon v \equiv 4, 6 \pmod 9), 12 \le v \le 247\}$.

Now, we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4 From Lemma 1.3, we need only to consider the values v, such that $v \in B_{10}, v \le 247$, the conclusion follows from Lemma 3.5 and Lemma 3.8. □

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组大小为 10 的广义 Steiner 三元系

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摘 要: 广义 Steiner 三元系 GS(2,3,n,g) 等价于 $g+1$ 元最优常重量码 $(n,3,3)$. 本文证明了 GS$(2,3,n,10)$ 存在的必要条件 $n \equiv 0,1(\text{mod}3), n \geq 12$ 也是充分的.