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On the Characteristics of the Convergence of Ishikawa Type Iterative Sequences for Strong Pseudocontractions and Strongly Accretive Operators *

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Abstract: In this paper, the results characterize the convergence of Ishikawa type iterative sequences (with errors) for constructing the solutions of strongly accretive operator equations, the solutions of m-accretive operator equations, and the fixed points of strong pseudocontractions. These results extend and improve Theorems 1-3 of Chidume and Osilike (Nonlinear Anal. TMA, 1999, 36(7): 863–872).

Key words: Ishikawa type iterative sequences; strong pseudocontractions; strongly accretive operators; arbitrary Banach spaces.

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1. Introduction

Let Ebe a real Banach space with norm $\|\cdot\|$ and dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*} that $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$, $x \in E$, where $\langle\cdot,\cdot\rangle$ is the generalized duality pairing.

An operator T with domain $D(T)$ and range $R(T)$ in E is called strongly accretive if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a constant k such that

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1}$$

In particular, T is said to be accretive if for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0. \tag{2}$$

Without loss of generality, we assume that $k \in (0,1)$. Closely related to the class of strongly accretive operators is the class of strong pseudocontractions where an operator T

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is called a strong pseudocontraction if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and t such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \frac{1}{t}\|x-y\|^2. \tag{3}$$

It is well known (see, e.g., [5]) that if $T: E \to E$ is continuous and strongly pseudo-contractive then T has a unique fixed point. Furthermore, if $T: E \to E$is continuous and strongly accretive, then T is surjective, i.e., for a given $f \in E$, the equation $Tx = f$ has a unique solution. Martin[4] has also proved that if $T: E \to E$ is continuous and accretive, then T is m-accretive so that the equation $x + Tx = f$ has a unique solution for any $f \in E$.

In 1995, Liu[9] introduced and studied the Ishikawa type iterative sequences with errors which are generated from an arbitrary $x_0 \in E$ by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \quad n \geq 0,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \quad n \geq 0,$$

where T is a mapping of E into itself, $\{u_n\}$ and $\{v_n\}$ are two sequences in E, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. It is obvious that if $u_n = v_n = 0, \forall n \geq 0$, then the above iteration method reduces to the Ishikawa iteration method.

Recently, Chidume and Osilike[8] extended the results of [10] from real smooth Banach spaces to arbitrary real Banach spaces.

Theorem CO1[8] *Suppose E is an arbitrary real Banach space and K is a nonempty closed convex bounded subset of E. Suppose $T: K \to K$ is a uniformly continuous strongly pseudocon traction and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions:*

(i) $0 \leq \alpha_n, \beta_n \leq 1, n \geq 0$, *(ii)* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, and *(iii)* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0,$$

converges strongly to the fixed point of T.

Theorem CO2[8] *Suppose E is an arbitrary real Banach space and $T: E \to E$ is a uniformly continuous strongly accretive operator. Suppose the range of $(I-T)$ is bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem CO1. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by*

$$y_n = (1-\beta_n)x_n + \beta_n(f + (I-T)x_n), \quad n \geq 0,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + (I-T)y_n), \quad n \geq 0$$

converges strongly to the unique solution of the equation$Tx = f$.

Theorem CO3[8] *Suppose E is an arbitrary real Banach space and $T: E \to E$ is a uniformly continuous accretive operator. Suppose the range of T is bounded and $\{\alpha_n\}$*

and $\{\beta_n\}$ *are as in Theorem CO1. Then the sequence* $\{x_n\}$ *generated from an arbitrary* $x_0 \in E$*by*

$$y_n = (1-\beta_n)x_n + \beta_n(f - Tx_n), \quad n \ge 0,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n), \quad n \ge 0$$

converges strongly to the solution of the equation $x + Tx = f$.

Motivated and inspired by the above results, we give the characteristic of the convergence of Ishikawa type iterative sequences (with errors) for constructing solutions of strongly accretive operator equations, solutions of m-accretive operator equations and fixed points of strong pseudo contractions. Our results extend and improve the above three Theorems CO1–CO3 by removing the restriction of $\lim_{n\to\infty}\beta_n = 0$ in Theorems CO1–CO3, and the restrictions that K is closed and bounded in Theorem CO1, the range of $(I-T)$ is bounded in Theorem CO2, and the range of T is bounded in Theorem CO3. Moreover, it is also easy to see that our results are interesting improvements and significant extensions of the results obtained previously by many authors including Chidume[10], Deng and Ding[2], Chidume and Osilike[8], Zeng[6,7], Zhou[11], and Chang, Cho, Lee, et al[1], since our results hold in the setting of arbitrary real Banach spaces.

2. Main results

2.1 Iterative approximations for strong pseudocontractions

Theorem 1 *Suppose E is an arbitrary real Banach space and K is a nonempty convex subset of E. Suppose $T: K \to K$ is a uniformly continuous strong pseudocontraction with a fixed point $x^\bullet$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions:*
(i) $0 \le \alpha_n, \beta_n \le 1, n \ge 0$, *(ii)* $\lim_{n\to\infty}\alpha_n = 0$, *and (iii)* $\sum_{n=0}^{\infty}\alpha_n = \infty$.
Let the sequence $\{x_n\}_{n=0}^{\infty}$ *be generated from an arbitrary* $x_0 \in K$ *by*

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \ge 0, \tag{4}$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \quad n \ge 0. \tag{5}$$

Then, $\{x_n\}$ *converges strongly to* $x^\bullet$ *if and only if* $\{x_n\}$ *is bounded, and* $\{x_n - Tx_n\}$ *converges strongly to zero.*

Proof At first, it follows from the strong pseudocontractive condition of T that the fixed point $x^\bullet$ of T is unique. Suppose $\{x_n\}$ converges strongly to $x^\bullet$. Then $\{x_n\}$ is bounded. Since

$$\|x_n - Tx_n\| \le \|x_n - x^\bullet\| + \|Tx_n - x^\bullet\|,$$

it follows from the uniform continuity of T that $\{x_n - Tx_n\}$ converges strongly to zero.

Conversely, suppose that $\{x_n\}$ is bounded and $\{x_n - Tx_n\}$ converges strongly to zero. Then we assert that $\{Tx_n\}, \{y_n\}$, and $\{Ty_n\}$ are all bounded. Indeed, since $\|Tx_n\| \le \|Tx_n - x_n\| + \|x_n\|$, and

$$\|y_n\| \le \|y_n - x_n\| + \|x_n\| = \beta_n\|Tx_n - x_n\| + \|x_n\| \le \|Tx_n - x_n\| + \|x_n\|,$$

obviously, $\{Tx_n\}, \{y_n\}$ are bounded. Since$\|y_n - x_n\| = \beta_n\|Tx_n - x_n\|$, $\{y_n - x_n\}$ converges strongly to zero. It then follows from the uniform continuity of T that $\{Ty_n - Tx_n\}$

converges strongly to zero. Noticing that $\|Ty_n\| \le \|Ty_n - Tx_n\| + \|Tx_n\|$, we infer that $\{Ty_n\}$ is also bounded.

Next, we show that $\{x_n\}$ converges strongly to x^*. Indeed, since T is strongly pseudo-con tractive, it follows from Eq. (3) that for all $x, y \in K$ there exist $j(x-y) \in J(x-y)$ and a constant t such that

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \frac{(t-1)}{t}\|x-y\|^2. \tag{6}$$

Set $(t-1)/t = k$. Then from inequality (6) we get $\langle (I-T-kI)x-(I-T-kI)y, j(x-y)\rangle \ge 0$ and it follows from Kato[3] that

$$\|x-y\| \le \|x-y+r[(I-T-kI)x-(I-T-kI)y]\| \tag{7}$$

for all $x, y \in K$ and $r > 0$. Following the ideas of Chidume and Osilike[8], from Eq. (5) we obtain

$$\begin{aligned} x_n - x^* =&(1+\alpha_n)(x_{n+1}-x^*) + \alpha_n[(I-T-kI)x_{n+1} - (I-T-kI)x^*]- \\ &(1-k)\alpha_n(x_n - x^*) + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{8}$$

Hence,

$$\begin{aligned} \|x_n - x^*\| \ge&(1+\alpha_n)\|x_{n+1}-x^*\| - (1-k)\alpha_n\|x_n - x^*\| - \\ &(2-k)\alpha_n^2\|x_n - Ty_n\| - \alpha_n\|Tx_{n+1} - Ty_n\| \quad \text{(using Eq.(7))}. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\| \le&[1-k\alpha_n+\alpha_n^2]\|x_n - x^*\| + \\ &(2-k)\alpha_n^2\|x_n - Ty_n\| + \alpha_n\|Tx_{n+1} - Ty_n\|. \end{aligned} \tag{9}$$

Since the sequences $\{x_n\}, \{Ty_n\}$ are bounded, there exists $M > 0$ such that

$$\|x_{n+1} - x^*\| \le [1-k\alpha_n]\|x_n - x^*\| + (3-k)M\alpha_n^2 + \alpha_n\|Tx_{n+1} - Ty_n\|. \tag{10}$$

Observe that

$$\begin{aligned} \|x_{n+1} - y_n\| &\le (1-\alpha_n)\beta_n\|x_n - Tx_n\| + \alpha_n\|Ty_n - y_n\| \\ &\le \|x_n - Tx_n\| + \alpha_n\|Ty_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $\{y_n\}, \{Ty_n\}$ are bounded. Hence, it follows from the uniform continuity of T that

$$\lim_{n\to\infty} \|Tx_{n+1} - Ty_n\| = 0.$$

Set $k\alpha_n = \delta_n$, and $(3-k)M\alpha_n^2 + \alpha_n\|Tx_{n+1} - Ty_n\| = \sigma_n$ in Eq.(10) to obtain

$$\|x_{n+1} - x^*\| \le [1-\delta_n]\|x_n - x^*\| + \sigma_n.$$

Clearly, $\sum_{n=0}^{\infty} \delta_n = \infty$, and $\sigma_n = o(\delta_n)$ and an application of Lemma W now implies that $\|x_{n+1} - x^*\| \to 0$ as $n \to \infty$. This completes the proof of Theorem 1.

Remark 1 Theorem 1 improves and extends Theorem CO1 in the following ways: (1) Theorem 1 removes the restriction on the boundedness of K in Theorem CO1; (2) $\{\beta_n\}$ in Theorem 1 may be an arbitrary sequence in $[0,1]$ since the restriction $\lim_{n\to\infty}\beta_n = 0$ in Theorem CO1 is removed; (3) Theorem 1 establishes the necessary and sufficient condition for the convergence of Ishikawa type iterative sequence. In addition, it is also readily seen that if in Theorem 1 the sequence $\{\beta_n\}$ still satisfies $\lim_{n\to\infty}\beta_n = 0$, then "$\lim_{n\to\infty}\|x_n - x^*\| = 0 \Leftrightarrow \{x_n\}, \{Tx_n\}$ are both bounded."

2.2 Iterative solutions for the strongly accretive operator equation $Tx = f$

In this section, $k \in (0,1)$ is the constant appearing in the definition of strongly accretive operators.

Theorem 2 *Suppose E is an arbitrary real Banach space and $T : E \to E$ is a uniformly continuous strongly accretive operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 1. Let $\{u_n\}$ and $\{v_n\}$ be the sequences in E such that $\|u_n\| = o(\alpha_n)$ and $\|v_n\| \to 0$ as $n \to \infty$. Let the sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in E$ by*

$$y_n = (1-\beta_n)x_n + \beta_n(f + (I-T)x_n) + v_n, \quad n \geq 0,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + (I-T)y_n) + u_n, \quad n \geq 0.$$

Then, $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$ if and only if $\{x_n\}$ is bounded and $\{Tx_n\}$ converges strongly to f.

Proof The existence of a solution to the equation follows from Deimling[5] and the uniqueness follows from the strong accretivity condition of T. Define $S : E \to E$ by $Sx = f + (I-T)x$. Let x^* denote the solution. Then x^* is a fixed point of S and S is uniformly continuous. Furthermore,

$$\langle (I-S)x - (I-S)y, j(x-y)\rangle \geq k\|x-y\|^2, \quad \forall x, y \in E,$$

so that S is strongly pseudocontractive. It now follows as in the proof of Theorem 1 that

$$\begin{aligned}\|x_{n+1} - x^*\| \leq & [1 - k\alpha_n + \alpha_n^2]\|x_n - x^*\| + (2-k)\alpha_n^2\|x_n - Sy_n\| + \\ & \alpha_n\|Sx_{n+1} - Sy_n\| + [(2-k)\alpha_n + 1]\|u_n\|. \end{aligned} \tag{11}$$

Suppose that $\{x_n\}$ is bounded and $\{Tx_n\}$ converges strongly to f. Then it is readily seen that $\{x_n - Sx_n\}$ converges strongly to zero. Moreover, it follows as in the proof of Theorem 1 that $\{Sx_n\}, \{y_n\}$, and $\{Sy_n\}$ are all bounded. Observe that

$$\begin{aligned}\|x_{n+1} - y_n\| & \leq (1-\alpha_n)\|x_n - y_n\| + \alpha_n\|Sy_n - y_n\| + \|u_n\| \\ & \leq (1-\alpha_n)\beta_n\|x_n - Sx_n\| + (1-\alpha_n)\|v_n\| + \alpha_n\|Sy_n - y_n\| + \|u_n\| \\ & \leq \|x_n - Sx_n\| + \|v_n\| + \alpha_n\|Sy_n - y_n\| + \|u_n\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

so that it follows from the uniform continuity of S that $\lim_{n\to\infty}\|Sx_{n+1} - Sy_n\| = 0$. Since $\{x_n\}, \{Sy_n\}$ are bounded, there exists $M > 0$ such that

$$\|x_{n+1} - x^*\| \leq (1-k\alpha_n)\|x_n - x^*\| + (3-k)M\alpha_n^2 + \alpha_n\|Sx_{n+1} - Sy_n\| + (2\alpha_n + 1)\|u_n\|.$$

Now, it follows as in the proof of Theorem 1 that $\lim_{n\to\infty}\|x_{n+1}-x^*\|=0$.

Conversely, suppose $\{x_n\}$ converges strongly to the unique solution of the equation $Tx=f$. Then it follows as in the proof of Theorem 1 that $\{x_n\}$ is bounded and $\{x_n - Sx_n\}$ converges strongly to zero. Hence, by the definition of $S, \{Tx_n\}$ converges strongly to f.

Remark 2 It is readily seen that if in Theorem 2 the sequence $\{\beta_n\}$ still satisfies $\lim_{n\to\infty}\beta_n=0$, then "$\lim_{n\to\infty}\|x_n-x^*\|=0 \Leftrightarrow \{x_n\},\{Tx_n\}$ are both bounded."

2.3 Iterative solutions for the equation $x+Tx=f$ involving accretive operator

Theorem 3 *Suppose E is an arbitrary real Banach space and $T: E\to E$ is a uniformly continuous accretive operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 1. Let $\{u_n\}$ and $\{v_n\}$ be the sequences in E such that $\|u_n\|=o(\alpha_n)$ and $\|v_n\|\to 0$ as $n\to\infty$. Let the sequence $\{x_n\}$ be generated from an arbitrary $x_0\in E$ by*

$$y_n=(1-\beta_n)x_n+\beta_n(f-Tx_n)+v_n, \quad n\geq 0,$$

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n(f-Ty_n)+u_n, \quad n\geq 0.$$

Then, $\{x_n\}$ converges strongly to the unique solution of the equation $x+Tx=f$ if and only if $\{x_n\}$ is bounded and $\{x_n+Tx_n\}$ converges strongly to f.

Proof The existence of a solution to the equation follows from Martin[4] and the uniqueness follows from the accretive condition of T. Define $S: E\to E$ by $Sx=f-Tx$. Let x^* denote the solution of the equation $x+Tx=f$. Then x^* is a fixed point of S and S is uniformly continuous. Furthermore, $\langle (I-S)x-(I-S)y, j(x-y)\rangle \geq \|x-y\|^2, \forall x,y\in E$. Thus, as in the proof of Theorem 1 we obtain

$$\|x_{n+1}-x^*\|\leq[1-\alpha_n+\alpha_n^2]\|x_n-x\|+\alpha_n^2\|x_n-Sy_n\|+\alpha_n\|Sx_{n+1}-Sy_n\|+(\alpha_n+1)\|u_n\|.$$

The remainder of the proof is similar to that of Theorem 2. Thus, we omit it.

Remark 3 It is readily seen that if in Theorem 3 the sequence $\{\beta_n\}$ still satisfies $\lim_{n\to\infty}\beta_n=0$, then "$\lim_{n\to\infty}\|x_n-x^*\|=0 \Leftrightarrow \{x_n\},\{Tx_n\}$ are both bounded."

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关于强伪压缩算子与强增生算子的 Ishikawa 型迭代序列收敛性的特征

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摘 要: 本文结果表征了用于构造强增生算子方程解，m- 增生算子方程解及强伪压缩算子不动点的 (带误差的)Ishikawa 型迭代序列的收敛性. 推广与改进了 Chidume 与 Osilike 的定理 1, 定理 2 及定理 3 (Nonlinear Anal. TMA, 1999, 36 (7): 863–872).

关键词: Ishikawa 型迭代序列；强伪压缩算子；强增生算子；任意 Banach 空间.