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The Dense Fractal Sets with the Hausdorff Dimension *

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Abstract: In this paper, for any given s $(0 \le s \le 1)$, we construct a Cantor-type set E_s such that $\dim_H E_s = s$ and E_s is dense in $[0,1]$.

Key words: Cantor-type set; Hausdorff dimension; Hausdorff measure.

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In the text books of the theory of functions of a real variable, it is quite common to discuss the Cantor set in the interval $[0,1]$. It is a closed set which is nowhere dense in $[0,1]$. Its Lebesgue mensure is zero, but has a power of the continuum. From the fractal geometry we know that this Cantor set E is a fractal set, whose Hausdorff dimension $s = \frac{\log 2}{\log 3}$ and Hausdorff measure $\mathcal{H}^s(E) = 1$ in this special dimension s. In this paper, for any given s $(0 \le s \le 1)$, we will construct a Cantor-type set E_s such that $\dim_H E_s = s$ and E_s is dense in $[0,1]$. It is believed that this discussion is helpfull for studying the sets of real numbers in the context of theory of measure and fractal geometry.

We suppose the readers have been familiar with the Hausdorff measure, relative concepts can be refered to [1].

(1) For $0 < s < 1$, we construct the fractal set E_s in $[0,1]$ such that $\dim_H E_s = s$, E_s is dense in $[0,1], \mathcal{H}^s(E_s) = \infty$ and E_s is $\mathcal{H}^s$ σ-finite.

(a) Construct the set $E_{s,n,0} \subset [0, \frac{1}{n}]$ with $\mathcal{H}^s(E_{s,n,0}) = n^{-s}$.

We first construct two closed intervals $\triangle_0$ and $\triangle_2$ by removing an open interval $\triangle_1$ in $[0, \frac{1}{n}]$ such that $|\triangle_0| = |\triangle_2| = \frac{c}{n}$, where $c = e^{-\frac{\log 2}{s}}$(that is $2c^s = 1$) and $|I|$ denotes the length of interval I. Inductively, for $\triangle_\sigma, \sigma = \varepsilon_1 \cdots \varepsilon_k, \varepsilon_i = 0$ or $2, i = 1, \cdots, k$, two closed intervals $\triangle_{\sigma 0}$ and $\triangle_{\sigma 2}$ can be obtained from $\triangle_\sigma$ by removing an open interval $\triangle_{\sigma 1}$ so that $|\triangle_{\sigma 0}| = |\triangle_{\sigma 2}| = c|\triangle_\sigma|$. Let

$$E_{s,n,0} = \cap_{k=1}^{\infty} \cup_{\substack{\varepsilon_i = 0 or 2 \\ i=1,\cdots,k}} \triangle_{\varepsilon_1 \cdots \varepsilon_k},$$

which is called a simple Cantor set.

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The following formula is given in [2]. It is convenient to evaluate the Hausdorff measure with this formula.

The formula of the integral for evaluating the Hausdorff measure Suppose μ to be a Radon measure, E a Borel set and $\underline{D}_\mu \mathcal{H}^s(x) = \lim_{\delta\to 0}\inf_{\substack{x\in I\\ |I|<\delta}} \frac{|I|^s}{\mu(I)}$. If $\underline{D}_\mu \mathcal{H}^s(x) < \infty$ for each $x \in E$, then

$$\mathcal{H}^s(E) = \int_E \underline{D}_\mu \mathcal{H}^s \,\mathrm{d}\mu.$$

Now we evaluate $\mathcal{H}^s(E_{s,n,0})$ with this formula. When $\sigma = \varepsilon_1 \cdots \varepsilon_k, \varepsilon_i = 0$ or $2, i = 1, \cdots, k$, we use $\triangle^{(k)}$ for $\triangle_\sigma$, and define a function of sets by

$$\mu(\triangle^{(k)}) = n^{-1}2^{-k}.$$

Since

$$\begin{aligned}
|\triangle^{(k)}|^s &= (n^{-1}c^k)^s = n^{-s}2^{-k}, \\
|\triangle^{(k+1)}|^s &= (n^{-1}c^{k+1})^s = n^{-s}2^{-k-1}, \\
\mu(\triangle^{(k)}) &= n^{s-1}|\triangle^{(k)}|^s = n^{s-1}(c^{-1}|\triangle^{(k+1)}|)^s \\
&= n^{s-1}\cdot 2|\triangle^{(k+1)}|^s = 2\mu(\triangle^{(k+1)}),
\end{aligned}$$

μ is a mass distribution in $[0, \frac{1}{n}]$ whose support is $E_{s,n,0}$ (cf. Proposition 1.7 of [3]). Certainly, μ is a Radon measure.

In Example 1 of [2], we evaluate the Hausdorff measure of a simple Cantor set. By the method given in [2] we can compute $\underline{D}_\mu \mathcal{H}^s(x) = n^{1-s}$ for each $x \in (0, \frac{1}{n}) \cap E_{s,n,0}$. So we have

$$\mathcal{H}^s(E_{s,n,0}) = \int_0^{\frac{1}{n}} \underline{D}_\mu \mathcal{H}^s \mathrm{d}\mu = n^{-1}\cdot n^{1-s} = n^{-s}.$$

(b) In $[0,1]$ shift $E_{s,n,0}$ right with distances $\frac{i}{n}$, and call the shifted sets by $E_{s,n,i}, i = 1, \cdots, n-1$. Let

$$E_{s,n} = \bigcup_{i=0}^{n-1} E_{s,n,i}.$$

Then we have

$$\mathcal{H}^s(E_{s,n}) = \sum_{i=0}^{n-1} \mathcal{H}^s(E_{s,n,i}) = n^{1-s}.$$

(c) Let

$$E_s = \bigcup_{n=1}^{\infty} E_{s,n},$$

and E_s is the set we want. In fact, $\dim_H E_s = s$ because the measure of E_s is $\mathcal{H}^s$ σ-finite, and $\mathcal{H}^s(E_s) = \infty$ since $\mathcal{H}^s(E_s) \geq \mathcal{H}^s(E_{s,n}) = n^{1-s}$ for each n. Let $x \in [0,1], \delta$ be any positive number. If we choose n satisfying $\frac{1}{n} < \delta$, then $E_{s,n} \cap (x-\delta, x+\delta) \neq \varphi$, so E_s is dense in $[0,1]$.

(2) Construct a fractal set E_0 in $[0,1]$ such that $dim_H E_0 = 0$, E_0 is dense in $[0,1]$ but is uncountable.

(a) Construct $E_{0,n,0} \subset [0, \frac{1}{n}]$ so that $\dim_H E_{0,n,0} = 0$ and $E_{0,n,0}$ is uncountable.

Let $\{c_k\}$ be a decreasing sequence of positive numbers satisfying $c_1 < \frac{1}{2}, c_k \to 0$ $(k \to \infty)$. By the simlar method, but requiring $|\triangle^{(k)}| = c_k|\triangle^{(k-1)}|$, we can construct a homogeneous Cantor set $E_{0,n,0}$, and as Example 2 of [2] we obtain

$$\dim_H E_{0,n,0} = \lim_{k\to\infty} \frac{-k\log 2}{\log(c_1c_2\cdots c_k)} = 0.$$

(b) Move $E_{0,n,0}$ right with distance $\frac{i}{n}$, and call these sets by $E_{0,n,i}, i = 1, \cdots, n-1$. Let

$$E_0 = \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{n-1} E_{0,n,i},$$

and E_0 is the set we want to construct.

Remark Let $E_0^* = \mathbf{Q} \cap [0,1]$, where $\mathbf{Q}$ is a set of rational number. Then $\dim_H E_0^* = 0$, E_0^* is dense in $[0,1]$, but it is countable.

(3) Construct a fractal set E_1 in $[0,1]$ so that $\dim_H E_1 = 1$, E_1 is dense in $[0,1]$ and $\mathcal{L}(E_1) = 0$, where $\mathcal{L}$ denotes the Lebesgue measure.

Let $c_k = 2^{-1}e^{-\frac{1}{k}}, k = 1, 2, \cdots$, then $c_k \to \frac{1}{2}$, $(k \to \infty)$. With the same method, but requiring $|\triangle^{(k)}| = c_k|\triangle^{(k-1)}|$, we also obtain a homogeneous Cantor set $E_{1,n,0} \subset [0, \frac{1}{n}]$ and we have

$$\dim_H E_{1,n,0} = \lim_{k\to\infty} \frac{-k\log 2}{\log(c_1c_2\cdots c_k)} = 1.$$

Since $2^k c_1c_2\cdots c_k = e^{-1}\cdot e^{-\frac{1}{2}}\cdots e^{-\frac{1}{k}} = e^{-(1+\frac{1}{2}+\cdots+\frac{1}{k})}$, we have

$$\mathcal{L}(E_{1,n,0}) = \lim_{k\to\infty} \frac{1}{n} 2^k c_1c_2\cdots c_k = 0.$$

By the similar method, we can construct $E_{1,n,i}$ and E_1 successively, and E_1 is the set we need.

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稠密的 s 维分形集

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摘 要: 对任意给定的 $0 \le s \le 1$, 本文构造 Cantor 型集 E_s, 使 $\dim_H E_s = s$, 且 E_s 在 $[0,1]$ 内稠密.

关键词: Cantor 型集； Hausdorff 维数； Hausdorff 测度.