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Incorrect Results for E-Convex Functions and E-Convex Programming *

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Abstract: A class of functions and a sort of nonlinear programming called respectively E-convex functions and E-convex programming were presented and studied recently by Youness in [1]. In this paper, we point out the most results for E-convex functions and E-convex programming in [1] are not true by six counter examples.

Key words: Convex sets; convex functions; convex programming; generalized converxity.

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1. Definitions and some results in [1]

In this section, we recall the relatived definitions and results given in [1] which will be used in our study.

Definition 1 (Def. 2.1 in [1]) *A set $M \subseteq R^n$ is said to be an E-convex set, if there exists a map $E : R^n \to R^n$ such that*

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall x, y \in M, \forall \lambda \in [0,1]. \tag{1}$$

Proposition 1 *If set $M \subseteq R^n$ is an E-convex set, then $E(M) \subseteq M$.*

Definition 2 (Def. 3.1 in [1]) *A function $f : R^n \to R$ is said to be E-convex on a set $M \subseteq R^n$ if there is a map $E : R^n \to R^n$ such that M is an E-convex set and*

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall x, y \in M, \ \lambda \in [0,1]. \tag{2}$$

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Furthermore, if the inequality signs in formula (2) is strict for $E(x) \neq E(y)$ and $\lambda \in (0,1)$, then f is called strictly E-convex.

Remark 1 The definition of strict E-convexity in [1] is not clear and definite, since the conditions for formula (2) being strict were not given.

Definition 3 (Def. 3.2 in [1]) *Let $S \subseteq R^n \times R$ and $E : R^n \to R^n$. The set S is said to be E-convex if $(x, \alpha), (y, \beta) \in S$ imply*

$$(\lambda E(x) + (1-\lambda)E(y), \lambda\alpha + (1-\lambda)\beta) \in S, \ \forall \lambda \in [0,1]. \tag{3}$$

Definition 4 (See Section 4 in [1]) *The nonlinear programming problem*

$$(\mathrm{P}) \quad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & x \in M = \{x \in R^n : \ g_i(x) \leq 0, i \in I\}. \end{array}$$

is said to be an E-convex programming if there exists a map $E : R^n \to R^n$ such that the functions $f, g_i \ (i \in I) : R^n \to R$ all are E-convex functions on R^n.

Throughout this paper, Problem (P) is always assumed to be an E-convex programming.

Theorem 1 (Theorem 3.1 in [1]) *A numerical function f defined on an E-convex set $M \subseteq R^n$ is E-convex on M iff its E-epigraph $E - e(f)$ is E-convex on $R^n \times R$, where*

$$E - e(f) = \{(x, \alpha) : \ x \in M, \alpha \in R, f(E(x)) \leq \alpha\}.$$

Theorem 2 (Theorem 4.1 in [1]) *The feasible set M of (P) is an E-convex set.*

Theorem 3 (Theorem 4.2 in [1]) *Assume that $E(M)$ is convex and $\bar{x}$ is a solution of the following problem:*

$$(\mathrm{P}_E) \quad \min\{(f \circ E)(x) \mid \ x \in M\}.$$

Then $E(\bar{x})$ is a solution of Problem (P), where $(f \circ E)(x) = f(E(x))$.

Theorem 4 (Theorem 4.3 in [1]) *Let $E(M)$ be a convex set. If $x^o = E(z^o) \in E(M)$ is a local minimum of Problem (P) on M, then x^o is a global minimum of Problem (P) on M.*

Theorem 5 (Theorem 4.4 in [1]) *Assume that $E(M)$ is a convex and f is strictlyE-convex. Then, the solution of Problem (P_E) is unique.*

Theorem 6 (Theorem 4.5 in [1]) *Let $E(M)$ is a convex set and let $f \circ E, g_i \circ E \ (i \in I)$ all be differebtiable on M. Assume that (x^*, y^*) is a solution of the following problem:*

$$\left\{ \begin{array}{l} \nabla_x[(f \circ E)(x^*) + y^*(g_i \circ E)(x^*)] = 0, \\ y^*(g_i \circ E)(x^*) = 0, \ (g_i \circ E)(x^*) \leq 0, \ y^* \geq 0, \ \forall i \in I. \end{array} \right. \tag{4}$$

Then, $E(x^*)$ is an optimal solution of Problem (P).

Theorem 7 (Theorem 4.6 in [1]) *Let Ω be the set of optimal solution of (P). Then Ω is a convex set.*

2. Counter examples for the above theorems

In this Section, we give six examples to show that the seven theorems given above are incorrect.

Example 1 A counter example for the necessity of Theorem 1.

Consider set M and maps $E, f: R \to R$ defined as

$$M = R = (-\infty, +\infty), \quad E(x) = -x^2, \quad f(x) = \begin{cases} 1, & \text{if } x > 0; \\ -x, & \text{if } x \leq 0. \end{cases}$$

Then M is an E-convex set, and f is an E-convex function on M since

$$f(\lambda E(x) + (1-\lambda)E(y)) = \lambda f(E(x)) + (1-\lambda) f(E(y)), \ \forall x, y \in M, \forall \lambda \in [0,1].$$

But the E-epigraph

$$E - e(f) = \{(x, \alpha) : \ x \in M, \alpha \in R, f(E(x)) = x^2 \leq \alpha\},$$

is not E-convex on R^2, since for $(x,\alpha) = (1,1), (y,\beta) = (2,4) \in E - e(f)$ and $\lambda = \frac{1}{3}$, one has

$$(\ \lambda E(x) + (1-\lambda)E(y), \ \lambda\alpha + (1-\lambda)\beta\ ) = (-3, 3), \ (-3)^2 \not\leq 3,$$

that is $(\ \lambda E(x) + (1-\lambda)E(y), \ \lambda\alpha + (1-\lambda)\beta\ ) = (-3,3) \notin E - e(f)$.

Example 2 A counter example for the sufficiency of Theorem 1.

Let $M = [0,1] \subset R, f(x) = -x^2, E(x) = \sqrt{x}, x \in M$.

Then M is an E-convex set, and

$$E - e(f) = \{(x, \alpha) : \ 0 \leq x \leq 1, \alpha \in R, f(E(x)) = -x \leq \alpha\}.$$

At first, we prove that the set $E - e(f)$ is E-convex on $R \times R$ as follows. Let $(x,\alpha), (y,\beta) \in E - e(f), \lambda \in [0,1]$, then

$$-\sqrt{x} \leq -x \leq \alpha, \quad -\sqrt{y} \leq -y \leq \beta, \ \lambda E(x) + (1-\lambda)E(y) \in M,$$

$$(\ \lambda E(x) + (1-\lambda)E(y), \ \lambda\alpha + (1-\lambda)\beta\ ) = (\ \lambda\sqrt{x} + (1-\lambda)\sqrt{y}, \ \lambda\alpha + (1-\lambda)\beta\ ),$$

$$-\lambda\sqrt{x} - (1-\lambda)\sqrt{y} \leq \lambda\alpha + (1-\lambda)\beta.$$

Thus

$$(\ \lambda E(x) + (1-\lambda)E(y), \ \lambda\alpha + (1-\lambda)\beta\ ) \in E - e(f).$$

Therefore set $E - e(f)$ is E-convex on $R \times R$ from Definition 3.

Secondly, one can conclude the given function f is not E-convex on the E-convex set M, since for $x = 0 \in M, y = 1 \in M, \lambda = \frac{1}{2}$, one gets

$$f(\lambda E(x) + (1-\lambda)E(y)) = f(\frac{1}{2}) = -\frac{1}{4} > -\frac{1}{2} = \lambda f(E(x)) + (1-\lambda) f(E(y)).$$

Example 3 A counter example for Theorem 2, i.e., Theorem 4.1 in [1].

Let maps $g, E : R \to R$ be given as

$$g(x) = \begin{cases} x, & \text{if } x < 0; \\ 1 - x, & \text{if } x \geq 0, \end{cases} \qquad E(x) = |x|.$$

Then the function g is E-convex on R, see Fig. 1, since

$$g(\lambda E(x) + (1 - \lambda)E(y)) = \lambda g(E(x)) + (1 - \lambda)g(E(y)), \ \forall x, y \in R, \lambda \in [0, 1].$$

But the feasible set

$$M = \{x : \ g(x) \leq 0\} = (-\infty, 0) \cup [1, +\infty)$$

is not E-convex frome Proposition 1, since $E(M) = (0, +\infty) \not\subset M$.

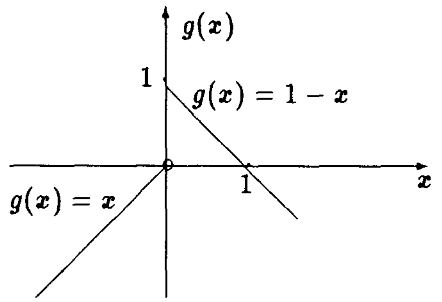

Fig.1 Example 3. A conuter example for Theorem 2, i.e., Theorem 4.1 in [1]

Example 4 A counter example for Theorems 3,4, i.e., Theorems 4.2, 4.3 in [1].

Consider functions $f, g, E : R \to R$ defined as

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1; \\ 2x, & \text{if } -1 < x < 0; \\ -x, & \text{if } 0 \leq x < 1; \\ -1, & \text{if } x \geq 1, \end{cases} \qquad g(x) = \begin{cases} 1 + x, & \text{if } x < 0; \\ 1 - x, & \text{if } x \geq 0, \end{cases} \qquad E(x) = |x|.$$

(i) Show that Thm. 3 is not true. See Fig. 2, one knows that

$$M = \{x : g(x) \leq 0\} = (-\infty, -1] \cup [1, +\infty), \ E(R) = [0, +\infty), \ E(M) = [1, +\infty),$$

$$(f \circ E)(x) = f(|x|) = \begin{cases} -|x|, & \text{if } 0 \leq |x| < 1; \\ -1, & \text{if } |x| \geq 1, \end{cases}$$

furthermore, f, g all are E-convex on R, $E(M)$ is a convex set .

It is clear that $\bar{x} = -2$ is a solution of (P_E), see Fig. 2, but $E(\bar{x}) = 2$ is not a solution of Problem (P) since $f(2) = -1 > f(-2) = -2$.

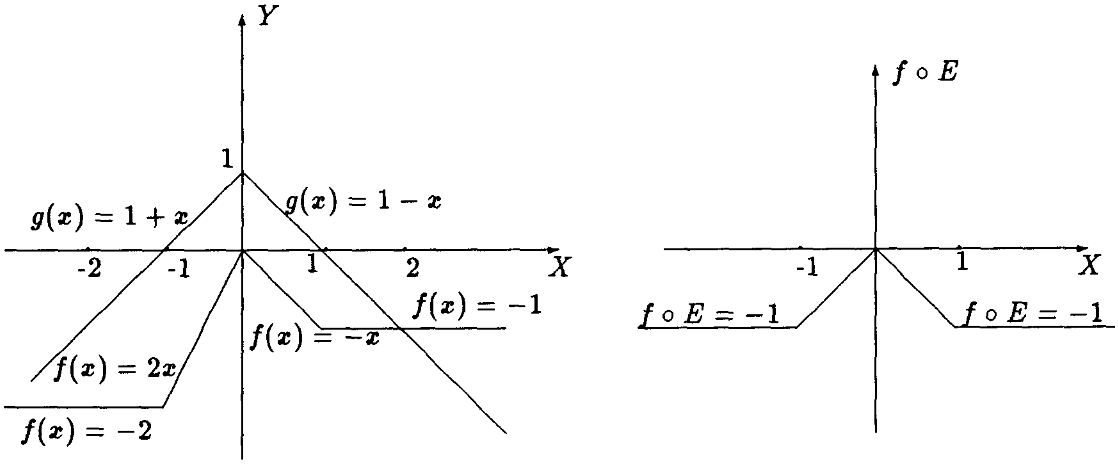

Fig.2 Example 4. A conuter example for Theorem 2, i.e., Theorem 4.1 in [1]

Example 5 A counter example for Theorems 5,6, i.e., Theorems 4.4, 4.5 in [1].

Let maps $f, g, E : R \to R$ be given as

$$f(x) = \begin{cases} -1, & \text{if } |x| \geq 1; \\ -|x|, & \text{if } |x| < 1, \end{cases} \quad g(x) = -\frac{1}{2}x, \quad E(x) \equiv 0.$$

Then, see Fig. 3, we know that functions f and g all are E-convex on R, and

$$M = \{x : \ g(x) \leq 0\} = [0, +\infty), \ E(M) = \{0\}; \ (f \circ E)(x) \equiv 0, \ (g \circ E)(x) \equiv 0, \ \forall x \in R.$$

$$\nabla_x[(f \circ E)(x)] = \nabla_x[(g \circ E)(x)] \equiv 0, \ \forall x \in R.$$

(i) Show that Theorem 5 is not true. From the above discusses , we know that $E(M)$ is a convex set, and f is strictly E-convex on R in the sense of Definition 2. But the solution of Problem (P_E) is not unique since each $x \in M$ is a solution of Problem (P_E).

(ii) Show that Theorem 6 is not true. In view of the above analyses, one knows that the functions $f \circ E, g \circ E$ all are differentiable on M, and each $(x^*, y^*) \in R^2$ with $y^* \geq 0$ is a solution of Problem (4), but $E(x^*) = 0$ is not a solution of Problem (P) since its optimal solution set $\Omega = [1, +\infty)$.

Remark 2 If the strict E-convexity of f on M is defined as

$$f(\lambda E(x) + (1 - \lambda)E(y)) < \lambda f(E(x)) + (1 - \lambda) f(E(y)), \ \forall x, y \in M, x \neq y, \lambda \in (0, 1),$$

then Theorem 5, i.e., Theorem 4.4 in [1] is true.

Example 6 A counter example for Theorem 7, i.e., Theorem 4.6 in [1].

Let maps $f, g, E : R \to R$ be given as

$$f(x) = \begin{cases} -4, & \text{if } |x| \geq 4; \\ -|x|, & \text{if } |x| < 4, \end{cases} \quad g(x) \equiv 0, \quad E(x) = \sqrt{|x|}.$$

Then, see Fig. 4, it is clear that functions f and g all are E-convex on R, and

$$M = \{x:\ g(x) \le 0\} = (-\infty, +\infty) = R,\ E(M) = E(R) = [0, +\infty).$$

But the optimal solution set $\Omega = (-\infty, -4] \cup [4, +\infty)$ is not a convex set. Furthermore Ω is not an E-convex set from Proposition 1, since $E(\Omega) = [2, +\infty) \not\subset \Omega$.

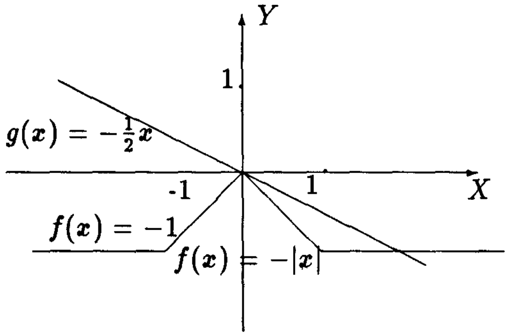

Fig.3 Example 5

A conuter example for Theorems 5, 6

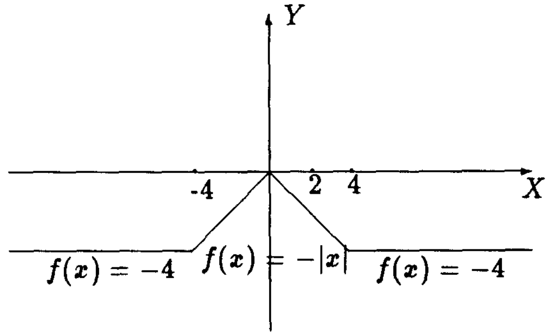

Fig.4 Example 6

A conuter example for Theorem 7

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关于 E- 凸函数和 E- 凸规划的错误结论

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摘 要: 最近 Youness 在文 [1] 建立了一类 E- 凸函数和一类 E- 凸规划，并分析和给出了他们的主要性质. 本文通过 6 个反例说明文 [1] 关于 E- 凸函数和 E- 凸规划的大部分结论是错误的.

关键词: 凸集；凸函数；凸规划；广义凸性.