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Are the Primitives of Fuzzy (K) Integrable Funcations Differentiable Almost Everywhere *

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Abstract: In this paper, an example is given. It shows that there exists a fuzzy-valued function which is (K) integrable on $[a,b]$, but its primitive is not differentiable almost everywhere in $[a,b]$.

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1. Notations and preliminaries

How to characterize the derivatives, in both of real analysis and fuzzy analysis, is an important problem. For a real-valued function, if it is Lebesgue integrable or Henstock integrable[2] on $[a,b]$, then the primitive is differentiable almost everywhere in $[a,b]$. Furthermore, the derivative equals the integrand function almost everywhere in $[a,b]$. For a fuzzy-valued function which is (K) integrable on $[a,b]$ (of course, it is fuzzy Henstock integrable[3]), the conclusion does not hold. In this paper, we give a fuzzy-valued function which is (K) integrable on $[0,1]$, but its primitive is not differentiable almost everywhere in $[0,1]$.

Let $\tilde{A} \in \tilde{F}(R)$ be a fuzzy subset on R. If $\tilde{A}$ is normal, convex, upper semicontinuous and has compact support, we say $\tilde{A}$ is a fuzzy number. Let $\tilde{R}^c$ denote the set of all fuzzy numbers[1,4].

For $\tilde{A}, \tilde{B} \in \tilde{R}^c$, the addition and scalar nultiplication are defined by the equations

$$[\tilde{A}+\tilde{B}]^r = [A]^r + [B]^r, \text{ i.e. } A_-^r + B_-^r = [\tilde{A}+\tilde{B}]_-^r \text{ and } A_+^r + B_+^r = [\tilde{A}+\tilde{B}]_+^r;$$

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$$[k\tilde{A}]^r = k[\tilde{A}]^r, \text{ i.e. } [k\tilde{A}]^r_- = \min\{kA^r_-, kA^r_+\} \text{ and } [kA]^r_- = \max\{kA^r_-, kA^r_+\};$$

respectively.

Define

$$D(\tilde{A},\tilde{B}) = \sup_{r\in[0,1]} d([A]^r,[B]^r) = \sup_{r\in[0,1]} \max\{|A^r_- - B^r_-|, |A^r_+ - B^r_+|\},$$

where d is Hausdorff metric. If we write $A^r = \{x : A(x) \geq r\}$, then $A^r = [A^r_-, A^r_+]$[1,3,4]. $D(\tilde{A},\tilde{B})$ is called the distance between $\tilde{A}$ and $\tilde{B}$.

Definition 1.1[1,4] *Let $\tilde{F} : [a,b] \to \tilde{R}^c$. Then (K) integral of $\tilde{F}(x)$ over $[a,b]$, denoted $(K)\int_a^b \tilde{F}(x)\mathrm{d}x$, is defined levelwise by the equation*

$$[\int_a^b \tilde{F}(x)\mathrm{d}x]^r = \{\int_a^b f(x)\mathrm{d}x : f : [a,b] \to R \text{ is a measurable selection for } F^r\}$$

for all $r \in [0,1]$.

A fuzzy-valued function $\tilde{F}$ is said to be strongly measurable on $[a,b]$ if F^r_- and F^r_+ are (L) measurable for all $r \in [0,1]$ (this definition is equivalent to the one used by Kaleva [1]).

A fuzzy-valued function $\tilde{F}$ is called integrably bounded if there exists an (L) integrable function h such that $|\dot{x}| \leq h(x)$ for all $\dot{x} \in F^r(x)$.

An integrably bounded and strongly measurable mapping is said to be (K) integrable on $[a,b]$ if the level set $\{[\int_a^b \tilde{F}(x)\mathrm{d}x]^r : r \in [0,1]\}$ determines a fuzzy number $\tilde{A} = \int_a^b \tilde{F}(x)\mathrm{d}x$.

In the sequel we denote $\int_a^b \tilde{F}(x)\mathrm{d}x$ as $(K)\int_a^b \tilde{F}(x)\mathrm{d}x$ and $\tilde{F} \in K[a,b]$.

Theorem 1.1[4] *Let $\tilde{F} : [a,b] \to \tilde{R}^c$ be a fuzzy-valued function. Then $\tilde{F} \in K[a,b]$ if and only if $F^r_-(x), F^r_+(x) \in L[a,b]$ for any $r \in [0,1]$. Furtheremore*

$$[(K)\int_a^b \tilde{F}(x)\mathrm{d}x]^r = [(L)\int_a^b F^r_-(x)\mathrm{d}x, (L)\int_a^b F^r_+(x)\mathrm{d}x].$$

Theorem 1.2[1] *Let $\tilde{F},\tilde{G} : [a,b] \to \tilde{R}^c$ be fuzzy-valued functions and $k \in R$. Then*

(1) $(K)\int_a^b (\tilde{F}+\tilde{G})\mathrm{d}x = (K)\int_a^b \tilde{F}\mathrm{d}x + (K)\int_a^b \tilde{G}\mathrm{d}x$;

(2) $(K)\int_a^b k\tilde{F}\mathrm{d}x = k(K)\int_a^b \tilde{F}\mathrm{d}x$;

(3) $D(\tilde{F},\tilde{G})$ *is (L) integrable, and*

$$D((K)\int_a^b \tilde{F}(x)\mathrm{d}x, (K)\int_a^b \tilde{G}(x)\mathrm{d}x) \leq (L)\int_a^b D(\tilde{F},\tilde{G})\mathrm{d}x;$$

(4) $\tilde{F}(x)$ *is (K) integrable on any subinterval of $[a,b]$.*

2. Differentiablility of the primitive

Definition 2.1[1,4] *Let $\tilde{F}:[a,b]\to\tilde{R}^c$. $\tilde{F}(x)$ is said to satisfy the condition (H) on $[a,b]$, if for any $x_1,x_2\in[a,b]$ satisfying $x_1<x_2$, there exists $\tilde{A}\in\tilde{R}^c$ such that*

$$\tilde{F}(x_2)=\tilde{F}(x_1)+\tilde{A}.$$

For brevity, we always assume the condition (H) is satisfied for the "$-$" operation of fuzzy numbers throughout this paper.

Definition 2.2[1,4] *A fuzzy-valued function $\tilde{F}:[a,b]\to\tilde{R}^c$ is said to be differentiable at $x_0\in[a,b]$, if there exists a $\tilde{F}'(x_0)$ such that the limits*

$$\lim_{h\to 0+}\frac{\tilde{F}(x_0+h)-\tilde{F}(x_0)}{h},\ \lim_{h\to 0+}\frac{\tilde{F}(x_0)-\tilde{F}(x_0-h)}{h}$$

exist and equal $\tilde{F}'(x_0)$. Here the limits is taken in the metric space $(\tilde{R}^c,D)$.

Remark 2.2.1[1] Let $\tilde{F}\in K[a,b]$ and $\tilde{G}(x)=(K)\int_a^x\tilde{F}(t)\mathrm{d}t$ be the primitive of $\tilde{F}(x)$ on $[a,b]$. Then $\tilde{G}(x)$ satisfies the condition (H).

Remark 2.2.2[1] Let $\tilde{F}:[a,b]\to\tilde{R}^c$ be continuous on $[a,b]$. Then, $\tilde{F}\in K[a,b]$ and

$$\tilde{G}'(x)=\tilde{F}(x),$$

where $\tilde{G}(x)=(K)\int_a^x\tilde{F}(t)\mathrm{d}t$ is the primitive of $\tilde{F}(x)$ on $[a,b]$.

Example 2.1 Define

$$\tilde{F}(x)=\begin{cases}0, & \text{if } s\notin[0,1],\\ x, & \text{if } s\in(0,1],\\ 1, & \text{if } s=0.\end{cases}$$

Then,

$$F_-^r(x)\equiv 0,$$

$$\tilde{F}_+^r(x)=\begin{cases}0, & \text{if } 0\le x<r,\\ 1, & \text{if } r\le x\le 1.\end{cases}$$

Note that $F_-^r(x), F_+^r(x)$ are (L) integrable for $r\in[0,1]$. From the theorem 1.1, $\tilde{F}\in K[a,b]$, and $\tilde{G}(x)=(K)\int_a^x\tilde{F}(t)\mathrm{d}t$ is determined by the equation

$$\tilde{G}^r(x)=[G_-^r(x),G_+^r(x)],\quad r\in[0,1],$$

where

$$G_-^r(x)\equiv 0,\quad G_+^r(x)=\begin{cases}0, & \text{if } 0\le x<r,\\ x-r, & \text{if } r\le x\le 1.\end{cases}$$

For any $x_0 \in [0,1]$ and $x > x_0$, we have

$$\begin{aligned}
D(\frac{\tilde{G}(x)-\tilde{G}(x_0)}{x-x_0}, \tilde{F}(x_0)) &= \frac{1}{x-x_0} \times D(\tilde{G}(x)-\tilde{G}(x_0), (x-x_0)\tilde{F}(x_0)) \\
&= \frac{1}{x-x_0} \times sup_{r\in[0,1]}\{|G_+^r(x) - G_+^r(x_0) - (x-x_0)F_+^r(x_0)|\} \\
&\geq \frac{1}{x-x_0} \times sup_{r\in(x_0,x]}\{|G_+^r(x) - G_+^r(x_0) - (x-x_0)F_+^r(x_0)|\} \\
&= \frac{1}{x-x_0} \times sup_{r\in(x_0,x]}\{|(x-r) - 0 - (x-x_0)0|\} = 1.
\end{aligned}$$

Lemma 2.1[1] *Let $\tilde{G}(x)$ be differentiable at $x \in [a,b]$. Then for any $r \in [0,1]$, $G_-^r(x), G_+^r(x)$ are differentiable at x. Furthermore*

$$[\tilde{G}'(x)]^r = [(G_-^r(x))', (G_+^r(x))'].$$

Theorem 2.1 *Let $\tilde{F} \in K[a,b]$, and $\tilde{G}(x) = (K)\int_a^x \tilde{F}(t)\mathrm{d}t$ be the primive. If $\tilde{G}(x)$ is differentiable almost everywhere, then*

$$\tilde{G}'(x) = \tilde{F}(x)$$

for almost all $x \in [a,b]$, i.e., if there exists subset $B \subset [a,b]$ with $L(B) = 0$ such that $\tilde{G}(x)$ is differentiable for any $x \in [a,b] \setminus B$, then there exists a subset $A \subset [a,b] \setminus B$ with $L(A) = 0$ such that $\tilde{G}'(x) = \tilde{F}(x)$ for any $x \in [a,b] \setminus (A \cup B)$.

Proof Since $\tilde{G}(x)$ is differentiable almost everywhere in $[a,b]$, there exists a subset $B \subset [a,b]$ with $L(B) = 0$ such that $\tilde{G}(x)$ is differentiable for any $x \in [a,b] \setminus B$. By lemma 2.1, for any $r \in [0,1]$, $G_-^r(x), G_+^r(x)$ are differentiable for any $x \in [a,b] \setminus B$, and

$$[\tilde{G}'(x)]^r = [(G_-^r(x))', (G_+^r(x))'].$$

That is,

$$[\tilde{G}'(x)]_-^r = (G_-^r(x))', \quad [\tilde{G}'(x)]_+^r = (G_+^r(x))'.$$

Since $\tilde{G}(x)$ is the primitive of $\tilde{F}(x)$ on $[a,b]$, for any $r \in [0,1]$,

$$(G_-^r(x))' = F_-^r(x) \text{ a.e } in[a,b], \quad (G_+^r(x))' = F_+^r(x) \text{ a.e } in[a,b].$$

This follows that there exists $A \subset [a,b] \setminus B$ with $L(A) = 0$ such that

$$[\tilde{G}'(x)]_-^r = (G_-^r(x))' = F_-^r(x), \quad [\tilde{G}'(x)]_+^r = (G_+^r(x))' = F_+^r(x)$$

for all $x \in [a,b] \setminus (A \cup B)$ and all rational numbers $r \in [0,1]$. By the left continuity of $[\tilde{G}'(x)]_-^r$, $[\tilde{G}'(x)]_+^r$, $F_-^r(x)$ and $F_+^r(x)$ at $r \in [0,1]$, we have

$$[\tilde{G}'(x)]_-^r = (G_-^r(x))' = F_-^r(x), \quad [\tilde{G}'(x)]_+^r = (G_+^r(x))' = F_-^r(x).$$

for all $x \in [a,b] \setminus (A \cup B)$ and all $r \in [0,1]$. It follows that

$$\tilde{G}'(x) = \tilde{F}(x) \text{ a.e } in[a,b].$$

This completes the proof.

By Theorem 2.1, we know that the primitive $\tilde{G}(x)$ of $\tilde{F}(x)$ on $[0,1]$ in Example 2.1 is not differentiable almost everywhere in $[a,b]$. That is, there exists a subset $E \subset [0,1]$ with $L(E) > 0$, $\tilde{G}(x)$ is not differentiable on E.

References:

[1] KLEVA O. *Fuzzy differential equations* [J]. Fuzzy Sets and Systems, 1987, **24**: 301–317.

[2] LEE P Y. *Lanzhou Lectures on Henstock Integration* [M]. World Scientific Publishing, Singapore, 1989.

[3] WU Cong-xin, GONG Zeng-tai. *On Henstock integral of interval functions and fuzzy-valued functions* [J]. Fuzzy Sets and Systems, 2000, **115**: 377–391.

[4] WU Cong-xin, MA Ming. *Embedding problem of fuzzy number space: Part II* [J]. Fuzzy Sets and Systems, 1992, **45**: 189–202.

模糊数值函数的积分原函数是否几乎处处可导

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摘 要: 对于模糊数值函数的积分原函数的可导性问题，本文构造性地给出一反例. 说明存在 (K) 可积的模糊数值函数其积分原函数并不是几乎处处可导的.

关键词: 模糊数；模糊数值函数；导数.