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A Version of Ekeland's Variational Principle in Countable Semi-Normed Spaces *

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Abstract: In this paper, a new version of Ekeland's variational principle in countable semi-normed spaces is given.

Key words: Ekeland's variational principle; topological vector space; countable semi-normed space.

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1. Introduction

Ekeland gave his famous variational principle for lower semicontinuous functions defined on complete metric spaces (where there are no linear structures), which plays very important role in nonlinear functional analysis (see [1], [2]). By using an $E \times R$ version of a classical maximality result due to Bishop and Phelps, Phelps obtained the following Ekeland's variational principle for lower semicontinuous functions defined on Banach spaces (see [3, p.47]):

Assume that f is a proper lower semicontinuous function on the Banach space E which is bounded below. Suppose that $\epsilon > 0$ and that $f(x_0) < \inf\{f(x) : x \in E\} + \epsilon$. Then for any λ with $0 < \lambda < 1$ there exists a point $z \in \mathrm{dom}(f)$ such that

(i) $\lambda\|z - x_0\| \le f(x_0) - f(z)$,

(ii) $\|z - x_0\| < \varepsilon/\lambda$,

(iii) $\lambda\|x - z\| + f(x) > f(z)$ whenever $x \neq z$.

Here a function $f : E \to R \cup \{+\infty\}$ is said to be proper if its effective domain, i.e., $\mathrm{dom}(f) = \{x \in E : f(x) < +\infty\}$, is nonempty.

In this paper we shall extend the above result to countable semi-normed spaces and obtain a new version of Ekeland's variational principle. First we need to make some preparations. Let $(E, \mathcal{T})$ be a complete topological vector space whose topology is generated by

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a sequence of semi-norms $\| \|_1 \leq \| \|_2 \leq \| \|_3 \leq \cdots$ (concerning topological vector spaces, see, for example, [4]). Inspired by [3, Definition 3.11], we define

$$K_{\lambda,n} = \{(x,r) \in E \times R : \lambda\|x\|_n \leq -r\}.$$

Then we have the following.

Lemma 1 *$K_{\lambda,n}$ is a closed convex cone in $E \times R$ and $K_{\lambda,1} \supset K_{\lambda,2} \supset \cdots$.*

The proof of Lemma 1 is routine and we omit it. The following Lemma 2 is the key to our main result.

Lemma 2 *Let A be a nonempty closed set in $E \times R$. Suppose that $0 < \lambda < 1$ and $\inf\{r : (x,r) \in A\} = 0$. Then for any $(x_0, r_0) \in A$ and $i \in N$, there exists $(z,r) \in A \cap (K_{\lambda,i} + (x_0, r_0))$ such that*

$$\{(z,r)\} = A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z,r)).$$

Proof Without loss of generality, we may assume that $i = 1$. Put $A_1 := A \cap (K_{\lambda,1} + (x_0, r_0))$. Then $A_1 \subset A$ and

$$\inf g(A_1) \geq \inf g(A) = \inf\{r : (x,r) \in A\} = 0,$$

where $g : E \times R \to R$ is defined as follows:

$$g(x,r) = r, \quad \forall (x,r) \in E \times R.$$

Choose $(x_1, r_1) \in A_1$ such that

$$r_1 < \inf g(A_1) + \frac{1}{2}.$$

Put $A_2 := A \cap (K_{\lambda,2} + (x_1, r_1))$, then $A_2 \subset A$ and

$$\inf g(A_2) \geq \inf g(A) = 0.$$

Choose $(x_2, r_2) \in A_2$ such that

$$r_2 < \inf g(A_2) + \frac{1}{2^2}.$$

In general, put

$$A_n := A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1}))$$

and choose $(x_n, r_n) \in A_n$ such that

$$r_n < \inf g(A_n) + \frac{1}{2^n}.$$

Repeating this process, we obtain a sequence $(x_n, r_n)_{n\in N}$ and a sequence $(A_n)_{n\in N}$ such that

$$(x_n, r_n) \in A_n = A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1})), \quad r_n < \inf g(A_n) + \frac{1}{2^n}, \quad \forall n \in N.$$

Obviously

$$(x_n, r_n) \in A_n \subset K_{\lambda,n} + (x_{n-1}, r_{n-1})$$

and by Lemma 1, $K_{\lambda,n+1} \subset K_{\lambda,n}$, hence we have

$$\begin{aligned} A_{n+1} &= A \cap (K_{\lambda,n+1} + (x_n, r_n)) \\ &\subset A \cap (K_{\lambda,n+1} + K_{\lambda,n} + (x_{n-1}, r_{n-1})) \\ &\subset A \cap (K_{\lambda,n} + K_{\lambda,n} + (x_{n-1}, r_{n-1})) \\ &= A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1})) \\ &= A_n. \end{aligned}$$

For $m \geq n+1$,

$$(x_m, r_m) \in A_m \subset A_{n+1} = A \cap (K_{\lambda,n+1} + (x_n, r_n)).$$

Hence

$$(x_m - x_n,\ r_m - r_n) \in K_{\lambda,n+1},$$

that is,

$$\lambda \|x_m - x_n\|_{n+1} \leq r_n - r_m.$$

Since $r_m \in g(A_m) \subset g(A_n)$, we have $r_m \geq \inf g(A_n)$. And since $r_n < \inf g(A_n) + 1/2^n$, we observe that

$$\lambda \|x_m - x_n\|_{n+1} \leq r_n - r_m < \inf g(A_n) + \frac{1}{2^n} - \inf g(A_n) = \frac{1}{2^n}.$$

Let $n_0 \in N$ be given. For $m \geq n+1 \geq n_0$, we have:

$$\lambda \|x_m - x_n\|_{n_0} \leq \lambda \|x_m - x_n\|_{n+1} < \frac{1}{2^n}, \quad 0 \leq r_n - r_m < \frac{1}{2^n}.$$

From this, we know that $(x_n)_{n\in N}$ is a Cauchy sequence in $(E, \mathcal{T})$ and $(r_n)_{n\in N}$ is a Cauchy real number sequence. Since $(E, \mathcal{T})$ and R are complete, there exist $z \in E$ and $r \in R$ such that $x_n \to x$ and $r_n \to r$, as $n \to \infty$.

Take arbitrarily $n \in N$, then clearly $A_n = A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1}))$ is closed. For every $m \geq n$, $(x_m, r_m) \in A_m \subset A_n$ and $(x_m, r_m) \to (z, r)$, as $m \to \infty$. Hence $(z, r) \in A_n$. Thus

$$\begin{aligned} (z, r) \in \bigcap_{n=1}^{\infty} A_n &= \bigcap_{n=1}^{\infty} (A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1}))) \\ &= A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (x_{n-1}, r_{n-1})). \end{aligned}$$

Particularly $(z,r) \in A_1 = A \cap (K_{\lambda,1} + (x_0, r_0))$. Next we show below that

$$\{(z,r)\} = A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z,r)).$$

It is easy to see that

$$(z,r) \in A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z,r)).$$

On the other hand, if

$$(z',r') \in A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z,r)),$$

then

$$\begin{aligned}(z',r') \in K_{\lambda,n} + (z,r) &\subset K_{\lambda,n} + A_{n+1} \\ &\subset K_{\lambda,n} + K_{\lambda,n+1} + (x_n, r_n) \subset K_{\lambda,n} + K_{\lambda,n} + (x_n, r_n) \\ &= K_{\lambda,n} + (x_n, r_n).\end{aligned}$$

From this, $(z' - x_n, r' - r_n) \in K_{\lambda,n}$, that is,

$$\lambda \|z' - x_n\|_n \le r_n - r'.$$

Remarking that

$$\begin{aligned}(z',r') \in K_{\lambda,n} + (z,r) &\subset K_{\lambda,n} + K_{\lambda,n} + (x_{n-1}, r_{n-1}) \\ &= K_{\lambda,n} + (x_{n-1}, r_{n-1})\end{aligned}$$

and $(z',r') \in A$, we conclude that

$$(z',r') \in A \cap (K_{\lambda,n} + (x_{n-1}, r_{n-1})) = A_n.$$

Therefore $r' \ge \inf g(A_n)$. Combining this with $r_n < \inf g(A_n) + 1/2^n$, we have:

$$\begin{aligned}\lambda \|z' - x_n\|_n &\le r_n - r' \\ &< \inf g(A_n) + \frac{1}{2^n} - \inf g(A_n) \\ &= \frac{1}{2^n}.\end{aligned}$$

Let $n_0 \in N$ be given. Then for all $n \ge n_0$ we have

$$\lambda \|z' - x_n\|_{n_0} \le \lambda \|z' - x_n\|_n < \frac{1}{2^n}, \quad 0 \le r_n - r' < \frac{1}{2^n}.$$

From this we see that $x_n \to z'$ in $(E, \mathcal{T})$ and $r_n \to r'$ in R. By the uniqueness of limits we conclude that $z' = z$ and $r' = r$, i.e., $(z',r') = (z,r)$. Thus we have proved that

$$\{(z,r)\} = A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z,r)). \quad \square$$

2. Ekeland's variational principle in countable semi-normed spaces

Now we come to our main result.

Theorem 1 *Let $(E, \mathcal{T})$ be a complete topological vector space whose topology is generated by a sequence of semi-norms $\|\ \|_1 \le \|\ \|_2 \le \|\ \|_3 \le \cdots$. Assume that $f : E \to R \cup \{+\infty\}$ is a proper lower semicontinuous function which is bounded below. Suppose that $x_0 \in E$ and $\varepsilon > 0$ such that $f(x_0) < \inf\{f(x) : x \in E\} + \varepsilon$. Then for any $0 < \lambda < 1$ and any $i \in N$, there exists $z \in \mathrm{dom}(f)$ such that*

(i) $\lambda\|z - x_0\|_i \le f(x_0) - f(z)$,

(ii) $\|z - x_0\|_i < \varepsilon/\lambda$,

(iii) $\lambda \lim_n \|x - z\|_n + f(x) > f(z),\ \forall x \ne z$.

Proof Since $f : E \to R \cup \{+\infty\}$ is lower semicontinuous and bounded below, we know that $A := \mathrm{epi}(f) = \{(x, r) : f(x) \le r\}$ is closed in $E \times R$ and $\inf f(E) > -\infty$. By the assumption, $f(x_0) < \inf f(E) + \varepsilon$. Without loss of generality, we may assume that $\inf f(E) = 0$. Thus $0 \le f(x_0) < \varepsilon$. Since $(x_0, f(x_0)) \in A$ and $\inf\{r : (x, r) \in A\} = 0$, by Lemma 2 we conclude that there exists $(z, r) \in A$ such that

$$(z, r) \in A \cap (K_{\lambda,i} + (x_0, f(x_0))) \tag{1}$$

and

$$\{(z, r)\} = A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z, r)). \tag{2}$$

From (1), we know that $0 \le f(z) \le r$ and $(z - x_0, r - f(x_0)) \in K_{\lambda,i}$, which means that

$$\lambda\|z - x_0\|_i \le f(x_0) - r \le f(x_0) - f(z) \le f(x_0) < \varepsilon.$$

Hence $\lambda\|z - x_0\|_i \le f(x_0) - f(z)$ and $\|z - x_0\|_i < \varepsilon/\lambda$. That is to say, the conclusions (i) and (ii) in Theorem 1 hold. We need yet to show that (iii) holds. First we show that $r = f(z)$, where (z, r) satisfies (2). Since $(z, r) \in A$, we always have that $f(z) \le r$. If $f(z) < r$, then $(z, r) \ne (z, f(z))$. Thus

$$(z, f(z)) \notin A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z, r)).$$

Remarking that $(z, f(z)) \in A$, we have

$$(z, f(z)) \notin \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z, r)).$$

Thus at least there exists $n \in N$ such that $(z, f(z)) \notin K_{\lambda,n} + (z, r)$, or $(0, f(z) - r) \notin K_{\lambda,n}$. That is, $\lambda\|0\|_n > r - f(z)$. This leads that $f(z) > r$, which contradicts the assumption that $f(z) < r$. Therefore we conclude that $r = f(z)$. That is,

$$\{(z, f(z))\} = A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z, f(z))). \tag{3}$$

If $x \neq z$ and $f(x) = +\infty$, certainly the conclusion (iii) holds. Now let us assume that $x \neq z$ and $f(x) < +\infty$. By (3) we know that

$$(x, f(x)) \notin A \cap \bigcap_{n=1}^{\infty} (K_{\lambda,n} + (z, f(z))).$$

Hence there exists $n = n(x) \in N$ such that

$$(x, f(x)) \notin K_{\lambda,n} + (z, f(z)).$$

From this,

$$(x - z, f(x) - f(z)) \notin K_{\lambda,n},$$

which means that

$$\lambda \|x - z\|_n > f(z) - f(x).$$

Hence

$$f(x) + \lambda \|x - z\|_n > f(z).$$

Since the sequence of semi-norms $(\| \|_n)_{n \in N}$ is increasing, we have

$$f(x) + \lambda \|x - z\|_m > f(z), \quad \forall m \geq n = n(x).$$

Hence

$$f(x) + \lambda \lim_n \|x - z\|_n > f(z),$$

which shows that the conclusion (iii) holds. □

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可数半范空间中 Ekeland 变分原理的一种形式

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摘 要: 本文给出了可数半范空间中 Ekeland 变分原理的一种新的形式.

关键词: Ekeland 变分原理；拓扑向量空间；可数半范空间.