Journal of Mathematical Research & Exposition
Vol.24, No.1, 13–17, February, 2004

Convergence of Nets in Induced $I(L)$-Topological Spaces with an Application *

WANG Ge-ping, CHEN Li
(Dept. of Math., Xuzhou Normal University, Jiangsu 221009, China)

Abstract: In this paper we give a characteristic property of convergence of nets in induced $I(L)$-topological spaces and a simplified proof for the N-compactness being an $I(L)$-"good extension".

Key words: Induced $I(L)$-ts; convergence of net; eventual α-net; N-compactness.

Classification: AMS(2000) 54D30, 54A40/CLC number: O189.1

Document code: A **Article ID:** 1000-341X(2004)01-0013-05

1. Basic Definitions

Let X be a non-empty set, L an F-lattice, i.e., a completely distributive lattice with an order-reversing involution $' : L \to L$, (L^X, δ) an L-topological space or an L-ts for short, and N the set of all natural numbers. For every $\alpha \in L$, $\underline{\alpha}$ denotes the fuzzy set taking the constant value α at any $x \in X$. An L-ts (L^X, δ) is called stratified, if for every $\alpha \in L$, $\underline{\alpha} \in \delta$. The sets of molecules in L and L^X are denoted by $M(L)$ and $M(L^X)$ respectively. For every $\alpha \in M(L)$, $\beta^*(\alpha)$ denotes the greatest minimum sets of α. For every $x \in X$ and $\alpha \in L$, the L-fuzzy point with support x and height α is denoted by x_α. It is clear that $x_\alpha \in M(L^X)$ if and only if $\alpha \in M(L)$. If D is a directed set, then $\{x^n_{\alpha_n}, n \in D\}$ is called a molecular net in (L^X, δ), where $x^n \in X$ and $\alpha_n \in M(L)$. We say that a molecular net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in (L^X, δ), if for every $P \in \delta'$ with $x_\alpha \notin P$, there exists $m \in D$ such that $x^n_{\alpha_n} \notin P$ for any $n \geq m$, where δ' denotes the set of all closed fuzzy sets in (L^X, δ). For every $a \in L$, denote $\downarrow a = \{b \in L : b \leq a\}$. The upper topology on L is the topology generated by $\{L - \downarrow a : a \in L\}$ as a subbase and denoted by Ω_L. The relative topology of Ω_L on $M(L)$ is denoted by Ω^*_L. Hence $(M(L), \Omega^*_L)$ is a topological space.

Let I denote the unit interval $[0,1]$. The L-fuzzy unit interval $I(L)$ is the set of all equivalence class $[\lambda]$, where $\lambda : R \to L$ is monotone decreasing mapping satisfying $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$, and $\mu \in [\lambda]$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for

*Received date: 2001-03-19
Foundation item: Supported by National Natural Science Foundation of China (10371079)
Biography: WANG Ge-ping (1941-), male, Professor.

all $t \in R$. The natural L-topology on $I(L)$ is generated from the subbase $\{L_t, R_t : t \in I\}$, where $L_t[\lambda] = \lambda(t-)'$ and $R_t[\lambda] = \lambda(t+)$. A partial order on $I(L)$ is naturally defined by $[\lambda] \leq [\mu]$ iff $\lambda(t-) \leq \mu(t-)$. Define $[\lambda] \wedge [\mu] = [\lambda \wedge \mu]$ and $[\lambda] \vee [\mu] = [\lambda \vee \mu]$. Moreover, let $\bar{\lambda} : R \to L$ satisfy $\bar{\lambda}(t) = \lambda(1-t)'$ for all $t \in R$ and define $[\lambda]' = [\bar{\lambda}]$. And then $(I(L), \vee, \wedge, ')$ is a completely distributive lattice with an order-reversing involution.

Lemma 1.1([5; Lemma 4.1]) *Let $\lambda \in I(L)$, then λ is a molecule in $I(L)$ iff there exists a molecule α in L and $t \in I$ such that $\lambda = \lambda_{\alpha,t}$, where*

$$\lambda_{\alpha,t}(s+) = \begin{cases} 1, & \text{if } s < 0, \\ \alpha, & \text{if } 0 \leq s < t, \\ 0, & \text{if } t \leq s. \end{cases}$$

Definition 1.1[5] *Let (L^X, δ) be an L-ts. A mapping $\mu : X \to I(L)$ is called I(L)-valued lower semicontinuous if $R_t\mu \in \delta$ for each $t \in I$. The set of all I(L)-valued lower semicontinuous mapping on X, being an I(L)-topology on X, is called an induced I(L)-topology which is denoted by $\omega(\delta)$. $(I(L)^X, \omega(\delta))$ is called an induced $I(L)$-topological space.*

Definition 1.2[8],[5] *Let $v \in L^X$. Define the characteristic function of v, denoted by χ_v, satisfying*

$$\chi_v(x)(t+) = \begin{cases} 1, & \text{if } t < 0, \\ v(x), & \text{if } 0 \leq t < 1, \\ 0, & \text{if } t \geq 1 \end{cases}$$

for any $x \in X$. Moreover, for any $t \in I$ define mapping $\hat{t} : X \to I(L)$ by

$$\hat{t}(x)(s+) = \begin{cases} 1, & \text{if } s < t, \\ 0, & \text{if } s \geq t \end{cases}$$

for any $x \in X$.

The definition of N-compactness in L-ts can be found in [1,2]. Let $\{x^n_{\alpha_n}, n \in D\}$ be a molecular net in L^X and $\alpha \in M(L)$. If for every $\lambda \in \beta^*(\alpha)$, there exists $m \in D$ such that $\alpha_n \geq \lambda$ for every $n \geq m$, then $\{x^n_{\alpha_n}, n \in D\}$ is called an eventual α-net. It has been proved that A is N-compact set in (L^X, δ) if and only if for every $\alpha \in M(L)$, every eventual α-net in A has a cluster point in A with height α, i.e., there exists a subnet converging to some $x_\alpha \in A$.

2. Main results

Lemma 2.1 *If the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$, then the net $\{\alpha_n, n \in D\}$ converges to α in $(M(L), \Omega^*_L)$ and for any positive real number $\varepsilon(\varepsilon < t)$, there exists an $m \in D$ such that $t_n > t - \varepsilon$ for every $n \geq m$.*

Proof Let $s = \sup\{t_n : n \in D\}$, $\beta = \sup\{\alpha_n : n \in D\}$. Suppose that $a \in L$ and $\alpha \in (L - \downarrow a) \cap M(L)$, then we have $\alpha \not\leq a$ and $\lambda_{\alpha,t} \not\leq \lambda_{a,s}$. Since $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$, there exists $m_1 \in D$ such that $\lambda_{\alpha_n,t_n} \not\leq \lambda_{a,s}$ for every $n \geq m_1$. Since $s = \sup\{t_n : n \in D\} \geq t_n$ for each $n \in D$, we get $\alpha_n \not\leq a$ for $n \geq m_1$. It follows that $\{\alpha_n, n \in D\}$ converges to α in $(M(I(L)), \Omega_L^*)$. If ε is a positive real number $(\varepsilon < t)$, then we have $\lambda_{\alpha,t} \not\leq \lambda_{\beta,t-\varepsilon}$. Since the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega_{I(L)}^*)$, there exists a $m_2 \in D$ such that $\lambda_{\alpha_n,t_n} \not\leq \lambda_{\beta,t-\varepsilon}$ for any $n \geq m_2$. Since $\beta = \sup\{\alpha_n : n \in D\} \geq \alpha_n$ for any $n \in D$, we get $t_n \geq t - \varepsilon$ for each $n \geq m_2$. This completes the proof.

Theorem 2.1 *Suppose that $(I(L)^X, \tau)$ is a stratified $I(L)$-ts. Let $[\tau] = \{U \in L^X : \chi_U \in \tau\}$(Obviously, $(L^X, [\tau])$ is a L-ts). If a molecular net $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \tau)$, then the net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in $(L^X, [\tau])$ and the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$.*

Proof Let $x_\alpha \not\leq U \in [\tau]'$. Then $\chi_{U'} \in \tau$ and $\chi_U \in \tau'$. Note that $x_{\lambda_{\alpha,t}} \not\leq \chi_U$. Since $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$, there exists $m \in D$ such that $x^n_{\lambda_{\alpha_n,t_n}} \not\leq \chi_U$ for every $n \geq m$. Hence $\alpha_n \not\leq \chi_U(x^n)(t_n-) = U(x^n)$, i.e., $x^n_{\alpha_n} \not\leq U$. It follows that $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in $(L^X, [\tau])$.

Suppose that V is a subbase open set in $(M(I(L)), \Omega^*_{I(L)})$ and $\lambda_{\alpha,t} \in V$. Then there exists $\lambda \in I(L)$ such that $V = (I(L) - \downarrow \lambda) \cap M(I(L))$. Since $\lambda_{\alpha,t} \in I(L) - \downarrow \lambda$, we have $\lambda_{\alpha,t} \not\leq \lambda$ and $x_{\lambda_{\alpha,t}} \not\leq \underline{\lambda}$. Note that $(L^X, [\tau])$ is a stratified L-ts, so $\underline{\lambda} \in \tau'$. It follows that there exists $m \in D$ such that if $n \geq m$, then $x^n_{\lambda_{\alpha_n,t_n}} \not\leq \underline{\lambda}$. So we have$\lambda_{\alpha_n,t_n} \not\leq \underline{\lambda}(x_n) = \lambda$, i.e., $\lambda_{\alpha_n,t_n} \in V$. Hence $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$.

The converse of Theorem 2. 1 is not true. it means that if a net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in $(L^X, [\tau])$ and a net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$, then the molecular net $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ may not converge to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \tau)$. We give a counterexample as follows:

Example 2.1 Let $X = \{x, y\}$ be a set with two points, $L = \{0, 1\}$. Then $(L^X, [\tau])$ is an ordinary topological space $(X, [\tau])$, $(I(L)^X, \tau)$ is a fuzzy topological space (I^X, τ). Define

$$\tau = \{\underline{a} : a \in I\} \cup \{x_\alpha \vee y_\beta : 0.5 \geq \alpha \geq \beta\}.$$

Then τ is a stratified fuzzy topology on I^X. It is easy to see that $[\tau]$ is a trivial topology on X. Denote a molecular net $\{x^n_{t_n}, n \in \mathrm{N}\}$ in I^X as follows: $t_n = 0.6$ for any $n \in \mathrm{N}$; $x^n = x$ whenever n is odd, $x^n = y$ whenever n is even. It is clear that $\{x^n : n \in \mathrm{N}\}$ converges to x. But $\{x^n_{t_n}, n \in \mathrm{N}\}$ does not converge to $x_{0.6}$ in (I^X, τ), because $x_{0.6} \notin x_{0.5} \vee y_1 \in \tau'$ and it is false that $\{x^n, n \in \mathrm{N}\}$ is eventually not in $x_{0.5} \vee y_1$.

Theorem 2.2 *Suppose that (L^X, δ) is a stratified L-ts, $(I(L)^X, \omega(\delta))$ is the induced $I(L)$-ts of (L^X, δ). Then a molecular net $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \omega(\delta))$ iff the net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in (L^X, δ) and the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$.*

Proof By Theorem 2.1, it is sufficient to prove the part of "if".

Suppose that$\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in (L^X, δ) and the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$. Since $\omega(\delta)$ has a base $\{\hat{s} \wedge \chi_U : s \in I, U \in \delta\}$, we can suppose that $P = \hat{s} \wedge \chi_U$ and $x_{\lambda_{\alpha,t}} \not\leq P'$. So $\lambda_{\alpha,t} \not\leq P'(x)$ and we get $\alpha \not\leq P'(x)(t-)$.

Hence $t > 1-s$ and $\alpha \not\leq U'(x)$. Since the net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α, there exists $m_1 \in D$ such that $x^n_{\alpha_n} \not\leq U'$ for every $n \geq m_1$. By Lemma 2.1, there exists $m_2 \in D$ such that $t_n > 1-s$ for every $n \geq m_2$. Since D is a directed set, there exists $m \in D$ such that $m \geq m_1$ and $m \geq m_2$. So we have $t_n > 1-s$ and $\alpha_n \not\leq U'(x^n)$ for every $n \geq m$. Hence we get $\alpha_n \not\leq U'(x^n) = P'(x^n)(t_n-)$ and $\lambda_{\alpha_n,t_n} \not\leq P'(x^n)$, i.e., $x^n_{\lambda_{\alpha_n,t_n}} \not\leq P'$. It follows that the net $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \omega(\delta))$.

Theorem 2.3 *Suppose that $(I(L)^X, \tau)$ is a $I(L)$-ts. If there exists an L-topology δ on L^X such that every molecular net $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \tau)$ iff the net $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in (L^X, δ) and the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$, then $(I(L)^X, \tau)$ is the induced $I(L)$-ts of (L^X, δ), that is $\tau = \omega(\delta)$.*

Proof Suppose that $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ is a molecular net converging to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \tau)$. By the assumption of Theorem, $\{x^n_{\alpha_n}, n \in D\}$ converges to x_α in (L^X, δ) and the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$. By Theorem 2.2, $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \omega(\delta))$. It follows from Theorem 5.1.17 in [4] that the identify mapping $f : (I(L)^X, \tau) \to (I(L)^X, \omega(\delta))$ is continuous. Hence $\omega(\delta) \subset \tau$. In a similar way, we get the identify mapping $g : (I(L)^X, \omega(\delta)) \to (I(L)^X, \tau)$ is continuous. So $\tau \subset \omega(\delta)$. Finally we get $\omega(\delta) = \tau$.

3. An application

As an application of Theorem 2.3, we will give a simplified proof which shows the N-compactness is an $I(L)$-"good extension" (see [5; Theorem 4.1]). At first we need the following lemma.

Lemma 3.1 *If $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ is an eventual $\lambda_{\alpha,t}$-net in $I(L)^X$, then the net $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$.*

Proof Let $\lambda \in I(L)$ and $\lambda_{\alpha,t} \in (L- \downarrow \lambda) \cap M(I(L)) \in \Omega^*_{I(L)}$. By $\vee\beta^*(\lambda_{\alpha,t}) = \lambda_{\alpha,t}$ and $\lambda_{\alpha,t} \not\leq \lambda$, there exists $\lambda_{\beta,s} \in \beta^*(\lambda_{\alpha,t})$ such that $\lambda_{\beta,s} \not\leq \lambda$. Since $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ is an eventual $\lambda_{\alpha,t}$-net, there exists $m \in D$ such that $\lambda_{\alpha_n,t_n} \geq \lambda_{\beta,s}$ for every $n \geq m$. It follows that $\lambda_{\alpha_n,t_n} \not\leq \lambda$, that is $\lambda_{\alpha_n,t_n} \in (I(L)- \downarrow \lambda) \cap M(I(L))$. Hence $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$.

Theorem 3.1 *Let $(I(L)^X, \omega(\delta))$ be the induced $I(L)$-ts of (L^X, δ). Then $(I(L)^X, \omega(\delta))$ is N-compact iff (L^X, δ) is N-compact.*

Proof Suppose that (L^X, δ) is N-compact, $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ is an arbitrary eventual $\lambda_{\alpha,t}$-net in $I(L)^X$. By Lemma 3.1, $\{\lambda_{\alpha_n,t_n}, n \in D\}$ converges to $\lambda_{\alpha,t}$ in $(M(I(L)), \Omega^*_{I(L)})$. If $\beta \in \beta^*(\alpha)$, then we have $\lambda_{\beta,t} \in \beta^*(\lambda_{\alpha,t})$. So there exists $m \in D$ such that $\lambda_{\alpha_n,t_n} \geq \lambda_{\beta,t}$ for every $n \geq m$. Then we get $\alpha_n \geq \beta$. Hence $\{x^n_{\alpha_n}, n \in D\}$ is a α-net in L^X. Since (L^X, δ) is N-compact, $\{x^n_{\alpha_n}, n \in D\}$ has a subnet, denoted by $\{x^{n_i}_{\alpha_{n_i}}, i \in E\}$, converging to some $x_\alpha \in L^X$. By Theorem 2.2, $\{x^{n_i}_{\lambda_{\alpha_{n_i},t_{n_i}}}, i \in E\}$ converges to $x_{\lambda_{\alpha,t}}$ in $(I(L)^X, \omega(\delta))$.

Obviously $\{x^{n_i}_{\lambda_{\alpha_{n_i},t_{n_i}}}, i \in E\}$ is a subnet of $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$. Hence $(I(L)^X, \omega(\delta))$ is N-compact.

Conversely, suppose that $(I(L)^X, \omega(\delta))$ is N-compact and $\{x^n_{\alpha_n}, n \in D\}$ is an arbitrary eventual α-net. Let $t \in I - \{0,1\}$. If $\lambda_{\beta,s} \in \beta^*(\lambda_{\alpha,t})$, then $\beta \in \beta^*(\alpha)$ and $s \le t$. So there exists $m \in D$ such that $\beta \le \alpha_n$ for every $n \ge m$. Hence we get $\lambda_{\alpha_n,t} \ge \lambda_{\beta,s}$ for every $n \ge m$. By the arbitrariness of $\lambda_{\beta,s}$, we have that $\{x^n_{\lambda_{\alpha,t}}, n \in D\}$ is a eventual $\lambda_{\alpha,t}$-net. By the assumption, $\{x^n_{\lambda_{\alpha_n,t_n}}, n \in D\}$ has a subnet, denoted by $\{x^{n_i}_{\lambda_{\alpha_{n_i},t_{n_i}}}, t \in E\}$, converging to some $x_{\lambda_{\alpha,t}}$. By Theorem 2.2, $\{x^{n_i}_{\alpha_{n_i}}, i \in E\}$ converges to x_α in (L^X, δ). Obviously $\{x^{n_i}_{\alpha_{n_i}}, i \in E\}$ is a subnet of $\{x^n_{\alpha_n}, n \in D\}$. Hence (L^X, δ) is N-compact.

References:

[1] PENG Yu-wei. *N-compactness in L-fuzzy topological spaces* [J]. Acta Math. Sinica, 1986, **29**: 555-558. (in Chinese)

[2] ZHAO Dong-sheng. *The N-compactness in L-fuzzy topological spaces* [J]. J. Math. Anal. Appl., 1987, **128**: 64–79.

[3] WANG Guo-jun. *Theory of L-Fuzzy Topological Spaces* [M]. Shanxi Normal University Press, 1988. (in Chinese)

[4] LIU Ying-ming, LUO Mao-kang. *Fuzzy Topology* [M]. World Scientific, Singapore, 1997.

[5] WANG Ge-ping. *Induced* **I(L)***-fuzzy topological spaces* [J]. Fuzzy Sets and Systems, 1991, **43**: 69–80.

[6] CHEN Yi-xiang. *On compactness of induced $I(L)$-fuzzy topological space* [J]. Fuzzy Sets and Systems, 1997, **88**: 373–378.

[7] LI Yong-ming. *The characterizations of convergence of molecular nets for induced spaces* [J]. Journal of Shanxi Normal University, 1991, **19**(2): 1–6. (in Chinese)

[8] KUBIAK T. *On fuzzy topologies*[D]. Ph. D. Thesis, UAM, Poznan, 1985; Chapter 3:I(L)-fuzzy sets and $I(L)$-topological spaces.

诱导 $I(L)$- 拓扑空间中网的收敛性及其应用

王 戈 平， 陈 莉

(徐州师范大学数学系，江苏 徐州 221009)

摘 要: 本文给出了诱导 $I(L)$- 拓扑空间中网的收敛性的一个刻画，利用它得到了良紧性是 $I(L)$-“好的推广” 的一个简洁的证明.

关键词: 诱导 $I(L)$- 拓扑空间；网的收敛性；最终 α- 网；良紧性.