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On the Ratio Inequalities for Locally Square Integrable Martingales *

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Abstract: In this paper, we establish some ratio inequalities for locally square integrable martingales, and give some extensions of the related results for continuous local martingales.

Key words: stopping time; locally square integrable martingale; supermartingale; ratio inequality.

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Let $M=(M_t)_{t\geq 0}$ be a continous local martingale with $M_0=0$. Set $M_t^*=\mathrm{Sup}_{s\leq t}|M_s|$, and let $\langle M\rangle=(\langle M\rangle_t)_{t\geq 0}$ be the increasing process associated with M. It was proved by Fefferman, Gundy and Yor[6] that there exists a universal constant $C_{p,q}$ such that

$$E(\frac{M_\infty^{*q}}{\langle M\rangle_\infty^{p/2}})\leq C_{p,q}\ E(M_\infty^{*q-p})\ \text{ with }\ q>p. \tag{1}$$

Also, Kikuchi[1] showed that there exists a universal constant $C_{\alpha,p}$ such that

$$E[M_\infty^{*p}\exp(\frac{\alpha M_\infty^{*2}}{\langle M\rangle_\infty})]\leq C_{\alpha,p}E(M_\infty^{*p})\ \text{ with }\ p>0, \tag{2}$$

$$E[\langle M\rangle_\infty^{p/2}\exp(\frac{\alpha M_\infty^{*2}}{\langle M\rangle_\infty})]\leq C_{\alpha,p}E(\langle M\rangle_\infty^{p/2})\ \text{ with }\ p>0 \tag{3}$$

with $0\leq\alpha<\frac{1}{2}$, and these inequalities are no longer valid for any $p>0$ when $\alpha\geq\frac{1}{2}$.

$$E[M_\infty^{*p}\exp(\frac{\alpha\langle M\rangle_\infty}{M_\infty^{*2}})]\leq C_{\alpha,p}\ E(M_\infty^{*p})\ \text{ with }\ p>0, \tag{4}$$

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$$E[\langle M\rangle_\infty^{p/2}\exp(\frac{\alpha\langle M\rangle_\infty}{M_\infty^{*2}})] \le C_{\alpha,p}\ E(\langle M\rangle_\infty^{p/2}) \text{ with } p>0 \tag{5}$$

with $0 \le \alpha < \frac{\pi^2}{8}$, and these inequalities are no longer valid for any $p > o$ when $\alpha \ge \frac{\pi^2}{8}$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space with filtration $(\mathcal{F}_t)_{t\ge 0}$ satisfying usual conditions. Denote by $\mathcal{M}^2_{\text{loc},0}$ the collection of all locally square integrable martingales with $M_0 = 0$ based on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. For $M \in \mathcal{M}^2_{\text{loc},0}$, $[M]$ is the quadratic variation of M, $\langle M\rangle$ is the predictable quadratic variation of M, M^c is the continuous part of M and M^d is the jump part of M. $M_t^* = \sup_{s\le t}|M_s|$. There are examples showing that inequalities $(2)-(5)$ are not valid for locally square integrable martingales .

The aim of this paper is to establish adequate extensions of $(2)-(5)$ for locally square integrable martingales. We shall work within the framework of general martingale theory, see for instance He, Wang and Yan[2], thus we consider martingales with cadalag paths and use the standard notions as in [3]. For this purpose we need some lemmas.

Lemma 1[4] *Let X and Y be two nonnegative random variables such that*

$$P(X > \gamma\lambda, Y \le \lambda) \le c\exp[-a(\sqrt{\gamma}-b)^2]P(X>\lambda) \tag{6}$$

for every $\lambda > 0$ and $\gamma > 1$, where a and c are two positive constants and b is an arbitrary constant. If $0 < \alpha < a$ and $p > 0$, there exists a constant $C = C(a,b,c,\alpha,p)$ such that

$$E[X^p\exp(\alpha X/Y)] \le CE(X^p),\quad E[Y^p\exp(\alpha X/Y)] \le CE(Y^p).$$

Furthermore, $E(X^p)$ can be replaced by $E(Y^p)$ in the first inequality.

Lemma 2[5] *Suppose that $M \in \mathcal{M}^2_{\text{loc}}, M_0 = 0$. Then for all $\delta > 0$, $\beta > 1+\delta$ and $\lambda > 0$, the following inequality holds:*

$$P(M_\infty^* > \beta\lambda, [M]_\infty + \langle M^d\rangle_\infty \le \delta^2\lambda^2) \le 2\exp\{-\frac{(\beta-1-\delta)^2}{2\delta^2}\}P(M_\infty^*>\lambda). \tag{7}$$

Lemma 3 *Suppose that $M \in \mathcal{M}^2_{\text{loc}}, M_0 = 0$. Then for all $\delta > 0$, $\beta > 1+\delta$ and $\lambda > 0$, the following inequality holds:*

$$P([M]_\infty^{1/2} > \beta\lambda, 2M_\infty^* + \langle M^d\rangle_\infty^{1/2} \le \delta\lambda) \le \exp(\frac{3}{8})\exp\{-\frac{(\beta-1-\delta)^2}{2\delta^2}\}P([M]_\infty^{1/2}>\lambda). \tag{8}$$

Proof By Itô's formula we have $M_t^2 = 2\int_0^t M_{s-}\, dM_s + [M]_t$, write

$$E^- = \{(s,\omega) : \triangle M_s \cdot M_s \le 0\},\quad N_t = [M]_t - M_t^2,\quad C_t = \sum_{s\le t}[(\triangle N_s)^+]^2,$$

$$D_t = \{\sum_{s\le t}[(\triangle N_s)^-]^2\}_t^{(p)},\ \tilde{D}_t = \{\sum_{s\le t} I_{E^-(s)}[(\triangle M_s)^-]^2\}_t^{(p)},\ H_t = \langle N^c\rangle_t + C_t + D_t,$$

where $A^{(p)}$ is the dual predictable projection of A. By the Proposition (4.2.1) of Barlow, Jacka, and Yor[6]

$$Z = (Z_t)_{t\geq 0} = (\exp[N_t - \frac{H_t}{2}])_{t\geq 0}$$

is a suppermartingale. Since $N_t = -2\int_0^t M_{s-}\, dM_s$, we have

$$C_t = \sum_{s\leq t}[(\triangle N_s)^+]^2 \leq 4\sum_{s\leq t} M_{s-}^2 (\triangle M_s)^2, \quad \langle N^c\rangle_t + C_t \leq 4(M^*)_t^2\cdot [M]_t,$$

$$D_t = \{\sum_{s\leq t}[(\triangle N_s)^-]^2\}_t^{(p)} = 4\int_0^t M_{s-}^2\, d\widetilde{D}_s \leq 4(M^*)_t^2\cdot \widetilde{D}_t.$$

Now, on the set $([M]_\infty^{1/2} > \lambda, M_\infty^* + \langle M^d\rangle_\infty^{1/2} \leq k)$, we have, for any $u > 0$

$$\begin{aligned} Z_\infty^{uM} &= \exp\{u([M]_\infty - M_\infty^{*2}) - \frac{u^2}{2}(\langle N^c\rangle_\infty + C_\infty + D_\infty)\} \\ &\geq \exp\{u([M]_\infty - M_\infty^{*2}) - \frac{u^2}{2}4M_\infty^*([M]_\infty + \widetilde{D}_\infty)\} \\ &\geq \exp\{u([M]_\infty - M_\infty^{*2}) - 2u^2k^2([M]_\infty + \widetilde{D}_\infty)\} \\ &= \exp\{(u-2u^2k^2)[M]_\infty - (uM_\infty^{*2} + 2u^2k^2\widetilde{D}_\infty)\} \\ &\geq \exp\{(u-2u^2k^2)\lambda^2 - (uk^2 + 2u^2k^4)\}. \end{aligned}$$

For any $A \in \mathcal{F}_0$, denote by $Z_t^{uM} I_A = \exp\{uM_t - \frac{u^2}{2}H_t\}I_A$, then $Z^{uM}I_A = (Z_t^{uM}I_A)_{t\geq 0}$ is also a suppermartingale. By suppermartingale inequality, we have

$$\begin{aligned} &P([M]_\infty^{1/2} > \lambda, M_\infty^* + \langle M^d\rangle_\infty^{1/2} \leq k, A) \\ &\quad \leq P(\sup_{t\geq 0} Z_t^{uM} I_A \geq \exp\{(u-2u^2k^2)\lambda^2 - (uk^2 + 2u^2k^4)\}) \\ &\quad \leq \exp\{(-u+2u^2k^2)\lambda^2 + (uk^2 + 2u^2k^4)\}P(A). \end{aligned}$$

Taking $u = \frac{1}{4k^2}$, we have

$$\begin{aligned} &P([M]_\infty^{1/2} > \lambda, M_\infty^* + \langle M^d\rangle_\infty^{1/2} \leq k, A) \\ &\quad \leq \exp\{(-\frac{1}{4k^2} + \frac{2k^2}{16k^4})\lambda^2 + (\frac{1}{4} + \frac{1}{8})\}P(A) \\ &\quad \leq \exp(\frac{3}{8})\exp\{-\frac{\lambda^2}{8k^2}\}P(A). \end{aligned} \tag{9}$$

For any stopping time T, define

$$\widetilde{M}_t = (M_{T+t} - M_T)I(T < \infty), \quad \mathcal{G}_t = \mathcal{F}_{T+t}, \qquad t \geq 0.$$

Then $\widetilde{M} = \{\widetilde{M}_t, \mathcal{G}_t, t\geq 0\}$ is a locally square integrable martingale with $\widetilde{M}_0 = 0$, and we have

$$[\widetilde{M}]_\infty = ([M]_\infty - [M]_T)I(T < \infty),$$

$$\langle \widetilde{M}^d\rangle_\infty = (\langle M^d\rangle_\infty - \langle M^d\rangle_T) I(T<\infty),$$
$$\widetilde{M}^*_\infty \le 2M^*_\infty, \quad (T<\infty)\in \mathcal{G}_0.$$

From (8) we have

$$\begin{aligned}
&P([M]^{1/2}_\infty - [M]^{1/2}_T > \lambda, 2M^*_\infty + \langle M^d\rangle^{1/2}_\infty \le k, T<\infty) \\
&\le P([\widetilde{M}]^{1/2}_\infty > \lambda, \widetilde{M}^*_\infty + \langle \widetilde{M}^d\rangle^{1/2}_\infty \le k, T<\infty) \\
&\le \exp(\frac{3}{8})\exp\{-\frac{\lambda^2}{8k^2}\}P(T<\infty). \qquad (10)
\end{aligned}$$

Now, for each fixed $\lambda>0$, we define stopping time τ by

$$\tau = \inf\{t\ge 0, [M]^{1/2}_t > \lambda\}.$$

Then $[M]^{1/2}_{\tau-}\le\lambda$, and on the set $(2M^*_\infty + \langle M^d\rangle^{1/2}_\infty \le \delta\lambda)$. We have $\triangle[M]^{1/2}_\tau \le (\triangle M)^*_\tau \le \delta\lambda$. Note that $([M]^{1/2}_\infty > \beta\lambda)\subset([M]^{1/2}_\infty>\lambda) = (\tau<\infty)$, by (10) we have

$$\begin{aligned}
&P([M]^{1/2}_\infty > \beta\lambda, 2M^*_\infty + \langle M^d\rangle^{1/2}_\infty \le \delta\lambda) \\
&\le P([M]^{1/2}_\infty - [M]^{1/2}_{\tau-} - \triangle[M]^{1/2}_\tau > (\beta-1-\delta)\lambda, 2M^*_\infty + \langle M^d\rangle^{1/2}_\infty \le \delta\lambda, \tau<\infty) \\
&\le P([M]^{1/2}_\infty - [M]^{1/2}_\tau > (\beta-1-\delta)\lambda, 2M^*_\infty + \langle M^d\rangle^{1/2}_\infty \le \delta\lambda, \tau<\infty) \\
&\le \exp(\frac{3}{8})\exp\{-\frac{(\beta-1-\delta)^2}{8\delta^2}\}P(\tau<\infty). \\
&= \exp(\frac{3}{8})\exp\{-\frac{(\beta-1-\delta)^2}{8\delta^2}\}P([M]^{1/2}_\infty > \lambda).
\end{aligned}$$

So the proof of Lemma 3 is completed.

Theorem 1 *Let $M=\{M_t,\mathcal{F}_t, t\ge 0\}$ be a locally square integrable martingale with $M_0=0$. Then for any $0<\alpha<\frac{1}{2}$, $p>0$, there exists a universal constant $C_{\alpha,p}$ such that*

$$E[M^{*p}_\infty \exp(\frac{\alpha M^{*2}_\infty}{[M]_\infty + \langle M^d\rangle_\infty})] \le C_{\alpha,p}\ E(M^{*p}_\infty), \qquad (11)$$

$$E[([M]_\infty + \langle M^d\rangle_\infty)^{p/2}\exp(\frac{\alpha M^{*2}_\infty}{[M]_\infty + \langle M^d\rangle_\infty})] \le C_{\alpha,p}\ E([M]_\infty + \langle M^d\rangle_\infty)^{p/2}. \qquad (12)$$

Proof For every $\lambda>0$ and $\gamma>1$, notice that $(M^{*2}_\infty > \gamma\lambda)\subseteq(M^{*2}_\infty>\lambda)$. By lemma 2, we have

$$\begin{aligned}
&P(M^{*2}_\infty > \gamma\lambda, [M]_\infty + \langle M^d\rangle_\infty \le \lambda) \\
&\le P(M^*_\infty > \sqrt{\gamma\lambda}, [M]_\infty + \langle M^d\rangle_\infty \le \lambda) \\
&\le C\exp\{-\frac{(\sqrt{\gamma}-2)^2}{2}\}P(M^*_\infty > \sqrt{\lambda}) \\
&\le C\exp\{-\frac{(\sqrt{\gamma}-2)^2}{2}\}P(M^{*2}_\infty > \lambda). \qquad (13)
\end{aligned}$$

By (13) and Lemma 1, we complete the proof of Theorem 1.

If $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a continuous locally square integrable martingale, the inequalities (11) and (12) become (2) and (3). So we get adequate extensions of (2) and (3).

Theorem 2 *Let $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ be a locally square integrable martingale with $M_0 = 0$. Then for any $0 < \alpha < \frac{1}{8}$, $p > 0$, there exists a universal constant $C_{\alpha,p}$ such that*

$$E\{[M]_\infty^{p/2} \exp[\frac{\alpha[M]_\infty}{(2M_\infty^* + \langle M^d\rangle_\infty^{1/2})^2}]\} \leq C_{\alpha,p}\ E([M]_\infty^{p/2}), \tag{14}$$

$$E\{(2M_\infty^* + \langle M^d\rangle_\infty^{1/2})^p \exp[\frac{\alpha[M]_\infty}{(2M_\infty^* + \langle M^d\rangle_\infty^{1/2})^2}]\} \leq C_{\alpha,p}\ E\{(2M_\infty^* + \langle M^d\rangle_\infty^{1/2})^p\}. \tag{15}$$

Proof For every $\lambda > 0$ and $\gamma > 1$, by Lemma 3 , we have

$$\begin{aligned} &P\{[M]_\infty > \gamma\lambda, (2M_\infty^* + \langle M^d\rangle_\infty^{1/2})^2 \leq \lambda\} \\ &\leq P([M]_\infty^{1/2} > \sqrt{\gamma\lambda}, 2M_\infty^* + \langle M^d\rangle_\infty^{1/2} \leq \sqrt{\lambda}) \\ &\leq C\exp(\frac{3}{8})\exp\{-\frac{(\sqrt{\gamma}-2)^2}{8}\}P\{[M]_\infty^{1/2} > \sqrt{\lambda}\} \\ &\leq C\exp(\frac{3}{8})exp\{-\frac{(\sqrt{\gamma}-2)^2}{8}\}P\{[M]_\infty > \lambda\}. \end{aligned} \tag{16}$$

Then the theorem immediately follows from (16) and Lemma 1.

The inequalities (14) and (15) are partial extensions of (4) and (5) except the restriction of α.

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局部平方可积鞅的比值不等式

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摘 要: 本文对局部平方可积鞅建立了几个比值不等式，推广了连续鞅的相应结果.

关键词: 停时；局部平方可积鞅；上鞅；比值不等式.