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More on Convering Squares with Squares *

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Abstract: This paper improves some lower bounds of a function $f(x)$ introduced in the problem of covering squares with squares.

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In [1], H.L. Abbott and M.Katchalski considered a special covering problem about covering squares with squares. Let $S: s_i = x^i,\ 0 < x < 1,\ i = 0, 1, 2, \cdots, \{Q_i\}$ is a sequence of closed squares, where Q_i has s_i as its side length. Denote

$$f(x) = \sup\{a : Q \text{ is a square with side length } a \text{ and } Q \text{ can be covered by } \{Q_i\}\}.$$

In the covering the sides of Q_i are paralled to those of Q. [1] gives several theorems to evaluate $f(x)$. In this paper, we improve the lower bound of $f(x)$. First we list the related results in [1].

Proposition 1[1] *Let R be a rectangle with sides a and $b, a \le b$, and S be a collection of $n+1$ squares with sides $x_0 > x_1 > \cdots > x_n$. If*

$$\sum_{i=0}^{n} x_i^2 \ge ab + (a+b)x_0,$$

then the squares from S may be used to cover R.

Theorem 2[1] $f(x) > \sqrt{\frac{2-x^2}{1-x^2}} - 2 > \sqrt{\frac{1}{1-x^2}} - 2.$

We improve Theorem 2 in [1] as follows.

Theorem 1 $f(x) \ge \sqrt{\frac{2-x^2}{1-x^2}} - 1.$

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Proof Let $a = \sqrt{\frac{2-x^2}{1-x^2}} - 1$. Then

$$\sum_{i=0}^{\infty} x^{2i} = a^2 + 2a.$$

For any $\varepsilon > 0$, there exists a smallest integer $N(\varepsilon)$ satisfying

$$\sum_{i=0}^{N(\varepsilon)} x^{2i} \geq (a-\varepsilon)^2 + 2(a-\varepsilon).$$

From Proposition 1, we get that the square Q with side of length $a-\varepsilon$ can be covered by the square sequence $Q_0, Q_1, \cdots, Q_{N(\varepsilon)}$, where Q_i has side of length x^i. From the definition of $f(x)$ and the arbitrariness of ε, we have $f(x) \geq a$, that is

$$f(x) \geq \sqrt{\frac{2-x^2}{1-x^2}} - 1.$$

We give a theorem which improves Proposition 1.

Theorem 2 *Let Q be a rectangle with sides of length a and $b, a \leq b, x_0 > x_1 > \cdots > x_n > 0$. If*

$$\sum_{i=0}^{n} x_i^2 \geq ab + bx_0,$$

then the sequence of squares $\{Q_i\}$(Q_i has side of length $x_i, i = 0, 1, \cdots, n$) can cover Q.

Proof We prove by induction on n. When $n = 0, \sum_{i=0}^{n} x_i^2 \geq ab + bx_0$, that is $x_0^2 \geq ab + bx_0$, so $x_0(x_0 - b) \geq ab$, then $x_0 > b$, the proposition obviously holds.

Suppose that the proposition holds when the number of squares in the sequence $\{Q_i\}$ is less than $n+1$. We now consider the sequence with $n+1$ squares. When $x_0 \geq b$, clearly the result holds. When $x_0 < b$, let $l = \min\{k : \sum_{i=0}^{k} x_i \geq a\}$. Since

$$\sum_{i=0}^{n} x_i^2 \geq ab + bx_0,$$

we have

$$ax_0 < ab + bx_0 \leq \sum_{i=0}^{n} x_i^2 < x_0 \sum_{i=0}^{n} x_i.$$

So $\sum_{i=0}^{n} x_i \geq a$. This proves the existence of l. And we can prove that $l \neq n$. If $l = n$, then $\sum_{i=0}^{n-1} x_i < a$, so $\sum_{i=0}^{n-1} x_i^2 \leq x_0 \sum_{i=0}^{n-1} x_i < ax_0$, then $\sum_{i=0}^{n} x_i^2 < ax_0 + x_n^2 < 2ax_0$. While $\sum_{i=0}^{n} x_i^2 \geq ab + bx_0 > b\sum_{i=0}^{n-1} x_i + bx_0 > bx_0 + bx_0 \geq 2ax_0$, a contraction.

Now dissect Q into two rectangles: R_1 with sides of lengths a and x_l, and R_2 with sides of lengths a and $b - x_l$.

Clearly, from $\sum_{i=0}^{l} x_i \geq a$, we know that R_1 can be covered by $Q_0, Q_1, \cdots, Q_l$. Next we prove that R_2 can be covered by $Q_{l+1}, Q_{l+2}, \cdots, Q_n$. Since

$$\sum_{i=0}^{n} x^2 \geq ab + bx_0,$$

we know

$$\sum_{i=l+1}^{n} x_i^2 \geq ab + bx_0 - \sum_{i=0}^{l} x_i^2 \geq ab + bx_0 - x_0 \sum_{i=0}^{l-1} x_i - x_l^2$$
$$\geq ab + bx_0 - ax_0 - x_l^2 \geq ab + (b-a)x_l - x_l^2 > a(b - x_l) + (b - x_l)x_{l+1}.$$

That is

$$\sum_{i=l+1}^{n} x_i^2 \geq a(b - x_l) + (b - x_l)x_{l+1}.$$

By the induction hypothesis, R_2 can be covered by $Q_{l+1}, \cdots, Q_n$. Therefore Q can be covered by $Q_0, \cdots, Q_n$, and the conclusion is reached.

Let

$$a = \sqrt{\frac{1}{4} + \frac{1}{1-x^2}} - \frac{1}{2}.$$

Then

$$\frac{1}{1-x^2} = \sum_{i=0}^{\infty} x^{2i} = a^2 + a.$$

Similar to the proof of Theorem 1, we have

Theorem 3 $f(x) \geq \sqrt{\frac{1}{4} + \frac{1}{1-x^2}} - \frac{1}{2}$.

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关于正方形序列覆盖正方形的注记

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摘 要: 本文研究正方形序列覆盖正方形问题中涉及的一个函数的下界问题. 设 $\{Q_i\}$ 为闭正方形序列,

$$f(x) = \sup\{a : Q \text{ 为正方形，其边长为 } a, \{Q_i\} \text{ 覆盖 } Q\},$$

本文给出了 $f(x)$ 的若干下界.

关键词: 覆盖; 正方形; 闭正方形序列.