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$\aleph$-Spaces and *mssc*-Images of Metric Spaces *

GE Ying

(Dept. of Math., Suzhou University, Jiangsu 215006, China)

Abstract: In this paper, we give some characterizations of $\aleph$-spaces by *mssc*-images of metric spaces, and prove that a space X is an $\aleph$-space if and only if X is a sequence-covering (sequentially quotient) *mssc*-image of a metric space, which answer a conjecture on $\aleph$-spaces affirmatively.

Key words: $\aleph$-space; sequence-covering mapping; k-network; cs-network; cs^*-network.

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How can generalized metric spaces be characterize by mapping images of metric spaces? This is one of the key questions of P. Alexandroff Conjecture [1]. In [2], S.Lin introduced *mssc*-mappings and proved that a space X is an $\aleph$-space if and only if X is a compact-covering *mssc*-image of a metric space. Related to this result, recently Lin raised the following conjecture in a private letter to the author.

Conjecture $\aleph$-spaces can be characterized by certain sequence-covering *mssc*-images of metric spaces.

In this paper, we investigate structures of some sequence-covering *mssc*-images of metric spaces, and give some affirmative answers for the above conjecture. We prove that a space X is an $\aleph$-space if and only if X is a sequence-covering (pseudo-sequence-covering, subsequence-covering, sequentially quotient) *mssc*-image of a metric space.

Throughout this paper, all spaces are regular and T_1, and all mappings are continuous and onto. N and ω denote the set of all natural numbers and the first infinite ordinal respectively. $\{x_n\}$ denotes a sequence $x_1, x_2, \cdots, x_n, \cdots$ of points in a space and (x_n) denotes a point $(x_1, x_2, \cdots, x_n, \cdots)$ in a product space. Let A be a subset of a space, and $\bar{A}$ the closure of A. Let X be a space, $\mathcal{U}$ be a collection of subsets of X, and $f : X \to Y$ be a mapping. Then

$$f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}.$$

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For terms which are not defined here, refer to [3].

Definition 1 *Let $f : X \longrightarrow Y$ be a mapping. Assume that each convergent sequence in the following definitions contains its limit.*

(1) f is an mssc-mapping[2] if X is a subspace of the product space $\Pi_{n\in N}X_n$ with each X_n being a metric space, and for each $y \in Y$, there is a sequence $\{V_n\}$ of open neighborhoods of y in Y such that

$$\overline{p_n(f^{-1}(V_n))}$$

is a compact subset of X_n for each $n \in N$, where $p_n : \Pi_{n\in N}X_n \to X_n$ is the projection;

(2) f is a sequence-covering mapping[4] (pseudo-sequence-covering mapping[5]) if for each convergent sequence S in Y, there is a convergent sequence L (a compact subset K) in X such that

$$f(L) = S \quad (f(K) = S);$$

(3) f is a sequentially quotient mapping[6] (subsequence-covering mapping[7]) if for each convergent sequence S in Y, there is a convergent sequence L (a compact subset K) in X such that $f(L)$ $(f(K))$ is a subsequence of S.

Remark 1 The following implications are obvious[5].

sequence-covering mapping$\Longrightarrow$pseudo-sequence-covering (sequentially quotient) mapping $\Longrightarrow$subsequence-covering mapping.

Definition 2[3] *Let X be a space, and let $\mathcal{P}$ be a cover of X.*

(1) $\mathcal{P}$ is a network for X if whenever $x \in U$ with U is open in X, then

$$x \in P \subset U$$

for some $P \in \mathcal{P}$;

(2) $\mathcal{P}$ is a k-network for X if whenever a compact subset $K \subset U$ with U open in X, then

$$K \subset P \subset U$$

for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a cs-network for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then

$$\{x\} \cup \{x_n : n \geq m\} \subset P \subset U$$

for some $m \in N$ and some $P \in \mathcal{P}$;

(4) $\mathcal{P}$ is a cs-network for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then*

$$\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$$

for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

Remark 2[8] A space X is an $\aleph$-space if and only if X has a σ-locally finite cs-network.

Lemma 1 *Let $f: X \to Y$ be an mssc-mapping. Then there is a base $\mathcal{B}$ of X such that $f(\mathcal{B})$ is a σ-locally finite network of Y.*

Proof Since $f: X \to Y$ is an *mssc*-mapping, let $\{X_n : n \in N\}$ be the family of metric spaces which satisfies the condition of Definition 1(1). For each $n \in N$, X_n has a σ-locally finite base $\mathcal{P}_n$. Put

$$\mathcal{B}_n = \{X \cap (\cap_{i \leq n} p_i^{-1}(P_i)) : P_i \in \mathcal{P}_i, i \leq n\},\ \mathcal{B} = \cup_{n \in N} \mathcal{B}_n.$$

Then $\mathcal{B}$ is a base of X. It is easy to see that $f(\mathcal{B})$ is a network of Y. Let each $y \in Y$. For each $n \in N$, there is a sequence $\{V_n\}$ of open neighborhoods of y in Y such that $\overline{p_n(f^{-1}(V_n))}$ is a compact subset of X_n for each $n \in N$. Put $V = \cap_{i \leq n} V_i$, then V intersects at most finite members of $f(\mathcal{B}_n)$, hence $f(\mathcal{B}_n)$ is locally finite in Y. This proves that $f(\mathcal{B})$ is a σ-locally finite network of Y. □

Lemma 2[3] *If $\mathcal{P}$ is a σ-hereditarily closure-preserving cs^*-network of a space X, then $\mathcal{P}$ is a k-network of X.*

In [3], S. Lin proved a pseudo-sequence-covering mapping is a sequentially quotient mapping if the domain is a space in which points are a $G'_\delta s$ ([3, Proposition 2.1.17]). We point out pseudo-sequence-covering mapping can be relax to subsequence-covering mapping. That is, we have the following lemma.

Lemma 3 *Let $f: X \to Y$ be a subsequence-covering mapping, and points in X be $G'_\delta s$. Then f is a sequentially quotient mapping.*

Proof Let S be a sequence in Y, which converges to y. f is a subsequence-covering mapping, there is a compact subset K in X such that $f(K) = S'$ is a subsequence of S. Put $S' = \{y\} \cup \{y_n : n \in N\}$, then $\{y_n\}$ converges to y. Pick $x_n \in f^{-1}(y_n) \cap K$, then $\{x_n\} \subset K$. Notice that K is a compact subspace in which points are $G'_\delta s$. K is the first countable, so K is sequentially compact, thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges to $x \in f^{-1}(y)$. This proves that f is sequentially quotient. □

Lemma 4 *Let $f: X \to Y$ be a mapping, and $\{y_n\}$ be a sequence converging to y in Y. If $\{B_n\}$ is a decreasing network of some $x \in f^{-1}(y)$ in X, and $\{y_n\}$ is eventually in $f(B_n)$ for each $n \in N$, then there is a sequence $\{x_n\}$ converging to x such that each $x_n \in f^{-1}(y_n)$.*

Proof For each $k \in N$, as $\{x_n\}$ is eventually in $f(B_k)$, there is $n_k \in N$ such that $y_n \in f(B_k)$ for $n > n_k$, so $f^{-1}(y_n) \cap B_k \neq \varphi$. Without loss of generality, we can assume $1 < n_k < n_{k+1}$. For each $n \in N$, pick $x_n \in f^{-1}(y_n)$ if $n < n_1$, and pick $x_n \in f^{-1}(y_n) \cap B_k$ if $n_k \leq n < n_{k+1}$, then $x_n \in f^{-1}(y_n)$. It is not difficult to prove that $\{x_n\}$ converges to x. □

Theorem 5 *The following statements are equivalent for a space X:*

(1) X is an $\aleph$-space.

(2) X is a sequence-covering mssc-image of a metric space.

(3) X is a pseudo-sequence-covering mssc-image of a metric space.

(4) X is a subsequence-covering mssc-image of a metric space.

(5) X is a sequentially quotient mssc-image of a metric space.

Proof (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) is obvious. (4) $\Longrightarrow$ (5) from Lemma 3. We need only to prove that (1) $\Longrightarrow$ (2) and (5) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). Let X be an $\aleph$-space, and $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ be a cs-network for X, where each $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$ be a locally finite collection of closed subsets of X. Without loss of generality, we can suppose that each $\mathcal{P}_n$ is closed with respect to finite intersection and $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$. We can assume that $A'_n s$ are mutually disjoint, and each A_n is endowed the discrete topology. Put

$$Z = \{b = (\alpha_n) \in \Pi_{n\in N} A_n : \{P_{\alpha_n}\}$$

is a network of x_b in X for some $x_b \in X$, and $P_{\alpha_{n+1}} \subset P_{\alpha_n}\}$.

Then Z is a subspace of the Tychonoff product space $\Pi_{n\in N} A_n$ of the family $\{A_n : n \in N\}$ of metric spaces, so Z is a metric space. It is easy to see that $f : Z \to X$ defined by $f(b) = x_b$ is a mapping.

Claim 1. f is an $mssc$-mapping.

Let $x \in X$. For each $n \in N$, since $\mathcal{P}_n$ is locally finite, there is an open neighborhood V_n such that V_n intersects at most finite members of $f(\mathcal{P}_n)$. Put

$$B_n = \{\alpha \in A_n : V_n \cap P_\alpha \neq \varphi\}.$$

Then B_n is finite and $p_n f^{-1}(V_n) \subset B_n$, hence $\overline{p_n f^{-1}(V_n)}$ is a compact subset of A_n, so f is an $mssc$-mapping.

Claim 2. f is a sequence-covering mapping.

Let $S = \{x_n : n \in N\} \cup \{x\}$ be a sequence with its limit x. As $\mathcal{P}$ is a cs-network, and notice the supposition of $\mathcal{P}$, there is $z = (\alpha_n) \in \Pi_{n\in N} A_n$ such that $\{P_{\alpha_n} : n \in N\}$ is a decreasing network of x in X and S is eventually in P_{α_n} for each $n \in N$. Then $f(z) = x$. Put $Z_n = \{(\beta_k) \in Z : \beta_k = \alpha_k \text{ for } k \le n\}$, then $\{Z_n\}$ is a decreasing base of z in Z. Now we prove that $f(Z_n) = \cap_{k\le n} P_{\alpha_k}$ for each $n \in N$ as follows.

Let $b = (\beta_k) \in Z_n$. Then

$$f(b) \in \cap_{k\in N} P_{\beta_k} \subset \cap_{k\le n} P_{\alpha_k},$$

so $f(Z_n) \subset \cap_{k\le n} P_{\alpha_k}$. On the other hand, let $y \in \cap_{k\le n} P_{\alpha_k}$. Then there is $c' = (\gamma'_k) \in Z$ such that $f(c') = y$. For each $k \in N$, let $P_{\gamma_k} = P_{\gamma'_k} \cap P_{\alpha_n} \in \mathcal{P}_k$ if $k > n$, and put $\gamma_k = \alpha_k$ if $k \le n$. Put $c = (\gamma_k)$. It is easy to see that $c \in Z_n$ and $f(c) = y$, that is $y \in f(Z_n)$, so $\cap_{k\le n} P_{\alpha_k} \subset f(Z_n)$. Thus $f(Z_n) = \cap_{k\le n} P_{\alpha_k}$ for each $n \in N$.

As S is eventually in P_{α_n} for each $n \in N$, S is eventually in $\cap_{k\le n} P_{\alpha_k} = f(Z_n)$ for each $n \in N$. By Lemma 4, there is a sequence $\{z_n\}$ converging to z such that each $z_n \in f^{-1}(x_n)$, so f is a sequence-covering mapping.

By the above, f is a sequence-covering $mssc$-mapping.

(5) $\Longrightarrow$ (1). Let $f : Z \to X$ be a sequentially quotient $mssc$-mapping, and Z be a metric space. Then there is a base $\mathcal{B}$ for Z such that $f(\mathcal{B})$ a σ-locally finite network for X from Lemma 1. By Lemma 2, we need only to prove that $f(\mathcal{B})$ is a cs^*-network for X. Let $\{x_n\}$ be a sequence in X, which converges to a point $x \in U$ with U open in X. Since

f is sequentially quotient, there is a sequence $\{z_n\}$ converging to z in Z with $f(z_k) = x_{n_k}$ for each $k \in N$. Notice that $z \in f^{-1}(x) \subset f^{-1}(U)$ and $\mathcal{B}$ is a base for Z. There is $B \in \mathcal{B}$ such that

$$z \in B \subset f^{-1}(U),$$

so $\{z\} \cup \{z_k : k \geq m\} \subset B \subset f^{-1}(U)$ for some $m \in N$, thus

$$\{x\} \cup \{x_{n_k} : k \geq m\} \subset f(B) \subset ff^{-1}(U) = U$$

for some $m \in N$ and $f(B) \in f(\mathcal{B})$. This proves that $f(\mathcal{B})$ is a cs^*-network for X. □

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ℵ- 空间和度量空间的 $mssc$- 映象

葛 英

(苏州大学数学系， 江苏 苏州 215006)

摘 要: 本文用度量空间的 $mssc$- 映象给出了 ℵ- 空间一些刻画，证明了空间 X 是 ℵ- 空间当且仅当 X 是度量空间的序列覆盖 (序列商)$mssc$- 映象，肯定地回答了关于 ℵ- 空间的一个猜想.

关键词: ℵ- 空间； $mssc$- 映射；序列覆盖映射； k- 网； cs- 网； cs^*- 网.