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Generalized Frames and Frame Operators *

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Abstract: In this paper, we introduce and study the generalized frame in a separable Hilbert space H. Using operator-theoretic-methods, we give some conditions for a generalized frame to be a tight frame, a dual frame, or an independent frame in H. We also prove some results concerning generalized frame operators.

Key words: Hilbert space; generalized frame; frame operator.

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1. Introduction

In [1], G. Kaiser studied many properties of generalized frames and introduced a series of useful results. In [2], Cao Huai-xin studied some properties and results of discrete frames. In the paper, we study the generalized frame in a separable Hilbert space H and give some conditions for a generalized frame to be a tight frame, a dual frame or an independent frame in H. Associating a generalized frame operator $S = T_h^*T_h(T_h : H \to L^2(\mu))$ for a generalized frame h in H, we get a closed relationship between generalized frames and their operators, and give a general method to construct a new frame from a given generalized frame and operator in $B(H)$.

Let H be a separable Hilbert space and (M, S, μ) be a measure space. A generalized frame [1] in H indexed M is a family of vectors $h = \{h_m \in H : m \in M\}$. For every $f \in H$, if the function $\tilde{f} : M \to \mathbf{C}$ defined by $\tilde{f}(m) = \langle f, h_m\rangle$ is measurable, and there is a pair of constants $0 < A_h \le B_h < \infty$ such that

$$A_h\|f\|_H^2 \le \|\tilde{f}\|_{L^2(\mu)} \le B_h\|f\|_H^2, \quad \forall f \in H, \tag{1.1}$$

the vectors $\{h_m\}_{m\in M} \subseteq H$ are called frame vectors, (1.1) is called the frame condition, and A_h and B_h are called frame bounds. The function $\tilde{f}$ is called the transform of f with

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respect to the frame, the map $T_h f = \tilde{f}\ \forall f \in H$ is called the analyzing operator, and the adjoint $T_h^* : L^2(\mu) \to H$ of T_h is given by

$$(T_h^* g)(m) = \int_M g(m) h_m \mathrm{d}\mu(m), \ \forall g \in L^2(\mu). \tag{1.2}$$

Here the formula (1.2) holds in the "weak" sense in H, and T_h is clearly linear and bounded. If M is at most countable and μ is the counting measure, $h = \{h_m \in H : m \in M\}$ is called a discrete frame [3].

Let $\{h_m\}_{m\in M}$ be a generalized frame in H. If the frame bounds A_h and B_h are equal to each other, then the frame is called tight generalized frame. In this case, the frame condition reduces to $A_h\|f\|_H^2 = \|\tilde{f}\|_{L^2(\mu)}^2$. If $\{h_m\}_{m\in M}$ and $\{k_m\}_{m\in M}$ are two generalized frames in H and

$$f = \int_M \langle f, h_m\rangle k_m \mathrm{d}\mu(m) = \int_M \langle f, k_m\rangle h_m \mathrm{d}\mu(m), \quad \forall f \in H, \tag{1.3}$$

then we call k a dual generalized frame of h. In this case, h is also a dual generalized frame of k, and the pair $\{h, k\}$ is called a dual pair of generalized frames. A frame $\{h_m\}_{m\in M}$ in H is an independent generalized frame in H, if for $g \in L^2(\mu)$ satisfying $\int_M g(m) h_m \mathrm{d}\mu(m) = 0,\ \forall f \in H$, we have $g = 0$ a.e. on M.

2. Some properties of generalized frames

For convenience, we use the notation F_H, F_H^t, F_H^i to denote the sets of all generalized frame, tight generalized frame, independent generalized frame in H, respectively.

Proposition 2.1 *Let $h, k \in F_H$, and $[A_h, \|T_h\|^2] \cap [A_k, \|T_k\|^2] = \emptyset$. Then $h \pm k \in F_H$.*

Proof Without loss of generality, we suppose that $\|T_k\|^2 < A_h$. It suffices to prove that there two positive constants L and N such that

$$L\|f\|_H^2 \leq \|T_{h\pm k} f\|_{L^2(\mu)}^2 \leq N\|f\|_H^2, \ \forall f \in H.$$

Since $T_{h\pm k}$ is bounded, the constant N does exist. Put $L = A_h^{1/2} - \|T_k\|$, then $L > 0$, and we have

$$\|T_{h\pm k} f\| \geq \|T_h\| - \|T_k\| \geq A_h^{1/2}\|f\| - \|T_k\| \cdot \|f\| = L\|f\|.$$

Proposition 2.2 *Let $h, k \in F_H$. Then the pair $\{h, k\}$ is a dual pair of frames if and only if $T_h^* T_k = I$.*

Proof The necessity is clear. We only need to prove the sufficiency. So, we assume that $h, k \in F_H$, and $T_h^* T_k = I$. We get

$$f = T_h^* T_k f = T_h^* \langle f, k_m\rangle = \int_M \langle f, k_m\rangle h_m \mathrm{d}\mu(m), \quad \forall f \in H. \tag{2.1}$$

On the other hand, the condition $T_h^* T_k = I$ implies that $T_k^* T_h = I$, and hence

$$f = T_k^* T_h f = T_k^* \langle f, h_m\rangle = \int_M \langle f, h_m\rangle k_m \mathrm{d}\mu(m), \quad \forall f \in H. \tag{2.2}$$

Combining (2.1) with (2.2) implies (1.3). This proves the sufficiency.

Proposition 2.3 *Let* $h = \{h_m\}_{m\in M} \subseteq H$. *Then*

(1) $h \in F_H \Leftrightarrow T_h$ *is bounded below* $\Leftrightarrow T_h^* T_h$ *is invertible* ;

(2) $h \in F_H^t \Leftrightarrow T_h$ *is a scaled isometric,* i.e., $T_h = \alpha U$ *for a nonzero scalar* α *and some isometric* U;

(3) $h \in F_H^i \Leftrightarrow T_h$*is invertible,* i.e., $T_h^{-1} \in B(L^2(\mu), H)$.

Proof Using the formula (1.1) and the definition of tight generalized frames, the assertion (1) and (2) are clear. We only need to prove the assertion (3).

Necessarity. Let $h \in F_H^i$. Then it is clear that T_h is bounded below by the assertion (1). And from the definition of independent generalized frames, we see that $T_h^* \tilde{f} = 0$ implies $\tilde{f} = 0$, i.e.,Ker$T_h^* = \{0\}$, thus T_h is sujective. This shows that T_h is invertible.

Sufficiency. Let T_h be invertible. We have that

$$\|f\| = \|T_h^{-1} T_h f\| \le \|T_h^{-1}\| \cdot \|T_h f\|,$$

so $\|T_h f\| \ge \|T_h^{-1}\|^{-1}\|f\|, \forall f \in H$, i.e., T_h is bounded below. By the assertion (1) we get $h \in F_H$. Now suppose that there exists a nonzero function $g \in L^2(\mu)$ such that

$$\int_M g(m) h_m \mathrm{d}\mu(m) = 0, \ \forall f \in H,$$

then there is a nonzero function $g \in L^2(\mu)$ such that $T_h^* g = 0$, i.e., Ker$T_h^* \ne \{0\}$, which is a contradiction to the fact that T_h is invertible. Hence, $h \in F_H^i$.

3. Main results

For a given generalized frame $h = \{h_m\}_{m\in M}$, composing T_h with the adjoint operator T_h^*, we get the frame operator $S : H \to H$

$$Sf = T_h^* T_h f = \int_M \langle f, h_m\rangle h_m \mathrm{d}\mu(m), \ \forall f \in H. \tag{3.1}$$

Clearly, S is a linear operator on H. If $h = \{h_m\}_{m\in M}$is a generalized frame with the frame bounds A_h and B_h, then

$$A_h\|f\|_H^2 \le \langle Sf, f\rangle \le B_h\|f\|_H^2, \ \ \forall f \in H. \tag{3.2}$$

From the formula (3.1), we have

$$f = \int_M \langle f, h_m\rangle S^{-1} h_m \mathrm{d}\mu(m), \ \forall f \in H, \tag{3.3}$$

and

$$S^{-1} f = \int_M \langle f, S^{-1} h_m\rangle S^{-1} h_m \mathrm{d}\mu(m), \ \forall f \in H. \tag{3.4}$$

Applying the operator $S^{-1}$to the vectors $\{h_m\}_{m\in M}$ leads to a new family of vectors $\{S^{-1} h_m\}_{m\in M}$. Hence, we obtain the following property.

Proposition 3.1 *Let $h = \{h_m\}_{m\in M}$ be a family of vectors in H and the function $\langle f, h_m\rangle, m \in M$ be measurable. Define an operator S by*

$$Sf = \int_M \langle f, h_m\rangle h_m \mathrm{d}\mu(m), \forall f \in H.$$

Then the family of vectors $h = \{h_m\}_{m\in M}$ is a generalized frames of H if and only if S is a positive invertible operator in $B(H)$.

Proof Using the formula (3.2), the necessarity is clear. We only need to prove the sufficiency. Conversely, if the operator S is a positive invertible operator, then for all $f \in H$, we have

$$\triangle(S)\|f\|^2 \le \langle Sf, f\rangle \le \|S\| \cdot \|f\|^2.$$

Hence, the family of vectors $h = \{h_m\}_{m\in M}$is a generalized frames with the frame constants $\triangle(S)$ and $\|S\|$. Here $\triangle(S) = \inf\{\|Sf\| : f \in H, \|f\| = 1\}$ denotes the minimal module of S.

Suppose that a generalized frame is tight. We have

$$\langle Sf, f\rangle = \lambda\|f\|^2 \text{ for all } f \in H,$$

where λ is a scalar. So, in this case, $S = \lambda I$. Furthermore, we get the following assertion.

Theorem 3.2 *Let $\{h_m\}_{m\in M}$ be a generalized frame with the frame operator S. Then the following assertions are equivalent.*

(1) $S = I$;

(2) $\|f\|^2 = \int_M |\langle f, h_m\rangle|^2 \mathrm{d}\mu(m), \ \forall f \in H$;

(3) $\|f\|^2 = \int_M \left|\langle f, S^{-1}h_m\rangle\right|^2 \mathrm{d}\mu(m), \ \forall f \in H$.

Proof Using the formula (3.1), (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are evident.

(2) $\Rightarrow$ (1) If S is the frame operator of $\{h_m\}_{m\in M}$, for all $f \in H$, by the formula (3.4) we see that

$$\langle S^{-1}f, f\rangle = \int_M |\langle f, S^{-1}h_m\rangle|^2 \mathrm{d}\mu(m) = \|S^{-1}f\|^2 = \langle S^{-2}f, f\rangle.$$

Thus, $S^{-1} = S^{-2}$. This implies $S = I$.

(3) $\Rightarrow$ (1) The proof is similar to the proof of (2) $\Rightarrow$ (1). Using the formula (3.1), we see that

$$\langle Sf, f\rangle = \int_M |\langle f, h_m\rangle|^2 \mathrm{d}\mu(m) = \int_M |\langle Sf, S^{-1}h_m\rangle|^2 \mathrm{d}\mu(m)$$
$$= \|Sf\|^2 = \langle S^2 f, f\rangle.$$

So, $S = S^2$. This implies $S = I$.

Next, we will show a general method to get a new generalized frame from a given generalized frame and operator in $B(H)$.

Theorem 3.3 *Let $\{h_m\}_{m\in M} \in F_H$ and $V \in B(H)$. Set $k = \{Vh_m\}_{m\in M}$. Then $k \in F_H$*

if and only if the adjont operator V^* of V is bounded below, i.e., *there exists a positive constant δ such that*

$$\|V^* f\| \geq \delta \|f\|, \ \forall f \in H. \tag{3.5}$$

Proof Using the formula (3.5), we have

$$\|V^* f\|^2 \geq \delta^2 \|f\|^2, \ \forall f \in H. \tag{3.6}$$

By the formula (1.1) with the frame bound A_h and the formula (3.6), we obtain

$$\delta^2 A_h \|f\|^2 \leq A_h \|V^* f\|^2 \leq \int_M |\langle V^* f, h_m\rangle|^2 \mathrm{d}\mu(m)$$
$$= \int_M |\langle f, V h_m\rangle|^2 \mathrm{d}\mu(m), \ \forall f \in H,$$

A similar estimate with the frame bound B_h yields

$$\int_M |\langle f, V h_m\rangle|^2 \mathrm{d}\mu(m) = \int_M |\langle V^* f, h_m\rangle|^2 \mathrm{d}\mu(m) \leq B_h \|V^*\|^2 \cdot \|f\|^2, \ \forall f \in H.$$

It is clear that

$$\langle f, V h_m\rangle \in L^2(\mu), \ \text{ for all } \ f \in H.$$

Hence, we complete the sufficiency part of the proof.

Necessarity. We assume that $k = \{V h_m\}_{m\in M} \in F_H$ with frame bounds $0 < C_k \leq D_k < \infty$ so that

$$C_k \|f\|^2 \leq \int_M |\langle f, V h_m\rangle|^2 \mathrm{d}\mu(m) \leq D_k \|f\|^2, \ \forall f \in H. \tag{3.7}$$

Using the upper frame bound B_h of $\{h_m\}_{m\in M}$ and the left inequality of (3.7), we obtain

$$C_k \|f\|^2 \leq \int_M |\langle f, V h_m\rangle|^2 \mathrm{d}\mu(m) = \int_M |\langle V^* f, h_m\rangle|^2 \mathrm{d}\mu(m) \leq B_h \|V^* f\|^2, \forall f \in H,$$

where $\delta^2 = \frac{C_k}{B_h}$. This completes the proof.

Corollary 3.4 *Let $\{h_m\}_{m\in M}$ be a generalized frame with the frame operator S. Then the family of vectors $h' = \{S^{-\frac{1}{2}} h_m\}_{m\in M}$ is a tight frame with the frame bounds 1 and 1.*

Proof By the Theorem 3.3, we see that $\{S^{-\frac{1}{2}} h_m\}_{m\in M} \in F_H$. So, we only need to prove $\{S^{-\frac{1}{2}} h_m\}_{m\in M} \in F_H^t$. Set $T_{h'} f = \langle f, S^{-\frac{1}{2}} h_m\rangle, \forall f \in H$. Clearly, $T_{h'} f \in L^2(\mu)$. From the formula (3.3), we have

$$\langle f, f\rangle = \int_M |\langle f, S^{-\frac{1}{2}} h_m\rangle|^2 \mathrm{d}\mu(m) = \|T_{h'} f\|^2, \ \forall f \in H.$$

This implies

$$\|T_{h'} f\| = \|f\|, \forall f \in H,$$

i.e., $T_{h'}$ is an isometric, by Propsition 2.3, $\{S^{-\frac{1}{2}}h_m\}_{m\in M} \in F_H^t$. In addition, it is clear that the tight frame bounds are 1 and 1.

Corollary 3.5 *Let H_1 be a subspace of H, P an orthogonal projection from H onto H_1, and $\{h_m\}_{m\in M}$ a generalized frame of H. Then $\{Ph_m\}_{m\in M}$ is a generalized frame of H_1.*

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广义框架和框架算子

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摘 要: 本文研究了可分的 Hilbert 空间 H 中的广义框架, 应用算子论方法给出了广义框架是 H 中紧广义框架, 对偶广义框架, 独立广义框架的充要条件: 证明了有关广义框架算子的一些结果.

关键词: Hilbert 空间; 广义框架; 框架算子.