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A Refinement of the Weighted Hilbert Inequality *

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Abstract: In this paper it is shown that a refinement on the weighted Hilbert inequality for double series can be established by introducing a proper non-zero real number R_ω. The expression of R_ω is given by means of the positive definiteness of a Gram matrix.

Key words: double series; quadratic form; weighted Hilbert's inequality; weight function.

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1. Introduction

The inequality of the form

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 \leq \pi^2(\sum_{n=1}^{\infty} a_n^2)(\sum_{n=1}^{\infty} b_n^2) \tag{1}$$

is called Hilbert's double series theorem[1]. The equality contained in (1) holds if and only if $\{a_n\}$, or $\{b_n\}$, is a zero-sequence. In our previous paper[2] the weighted Hilbert inequality was established. To be specific, the inequality (1) can be generalized into the following form:

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 \leq (\sum_{n=1}^{\infty} \omega(n) a_n^2)(\sum_{n=1}^{\infty} \omega(n) b_n^2), \tag{1'}$$

where $\omega(n) = \pi - \frac{\theta(n)}{\sqrt{n}}$ (with $\theta(n) > 0$). Gao and Yang[3] proved that an infimum of $\theta(n)$ is $\theta(1) = \frac{\pi}{2} - \frac{7}{24} + \frac{\xi}{320}(0 < \xi < 1)$. However, if we select a proper non-zero real number R_ω, such that the right-hand side of (1') is replaced by $(\sum_{n=1}^{\infty}\omega(n)a_n^2)(\sum_{n=1}^{\infty}\omega(n)b_n^2) - R_\omega^2$, then some further results of the papers [2] and [3] will be obtained. The main purpose of the present paper is to prove the existence of the non-zero real number R_ω and to find an expression for it.

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For convenience, we first introduce some notations and define some functions.

The inner product of two vectors α and β in an inner space E is denoted by (α,β), and the norm of α is denoted by $\|\alpha\| = \sqrt{(\alpha,\alpha)}$. We next introduce a binary quadratic form defined by

$$F(x,y) = \|\alpha\|^2x^2 - 2(\alpha,\beta)xy + \|\beta\|^2y^2, \tag{2}$$

where $x = (\beta,\gamma)$ and $y = (\alpha,\gamma)$.

We further define

$$R = \|\alpha\||(\beta,\gamma)| - \|\beta\||(\alpha,\gamma)|. \tag{3}$$

The results involve R with α and β specified beforehand, and γ to be chosen for maximum felicity. It is obvious that, if $\|\alpha\||(\beta,\gamma)| = \|\beta\||(\alpha,\gamma)|$, then $R = 0$. Therefore, it is shrewd in every case to choose γ such that $\|\alpha\||(\beta,\gamma)| \neq \|\beta\||(\alpha,\gamma)|$.

2. Lemmas

In order to prove our assertions, we need the following lemmas.

Lemma 1 *Let $F(x,y)$ and R be functions defined respectively by (2) and (3). If $\|\alpha\||(\beta,\gamma)| \neq \|\beta\||(\alpha,\gamma)|$, then $F(x,y) \geq R^2 > 0$, and the equality holds if and only if $xy = 0$.*

Proof Applying Cauchy-Schwarz's inequality to estimate the right-hand side of (2) as follows:

$$\begin{aligned} F(x,y) &= \|\alpha\|^2x^2 - 2(\alpha,\beta)xy + \|\beta\|^2y^2 \\ &\geq \|\alpha\|^2x^2 - 2|(\alpha,\beta)xy| + \|\beta\|^2y^2 \geq (\|\alpha\||x| - \|\beta\||y|)^2. \end{aligned}$$

Since $\|\alpha\||(\beta,\gamma)| \neq \|\beta\||(\alpha,\gamma)|$, it follows that

$$F(x,y) \geq R^2 > 0. \tag{4}$$

Clearly, the equality in (4) holds if and only if $xy = 0$.

Lemma 2 *Let α,β and γ be three arbitrary vectors of the space E. If $\|\gamma\| = 1$ and $\|\alpha\||(\beta,\gamma)| \neq \|\beta\||(\alpha,\gamma)|$, then*

$$(\alpha,\beta)^2 \leq \|\alpha\|^2\|\beta\|^2 - R^2, \tag{5}$$

where R is a function defined by (3). And the equality contained in (5) holds if and only if the vector γ is a linear combination of α and β, and $xy = 0$.

Proof Consider the Gram determinant constructed by the vectors α,β and γ. According to the paper [4] we have

$$\begin{vmatrix} (\alpha,\alpha) & (\alpha,\beta) & (\alpha,\gamma) \\ (\beta,\alpha) & (\beta,\beta) & (\beta,\gamma) \\ (\gamma,\alpha) & (\gamma,\beta) & (\gamma,\gamma) \end{vmatrix} \geq 0. \tag{6}$$

The equality contained in (6) holds if and only if the vectors α,β and γ are linearly dependent. Since $\|\alpha\||(\beta,\gamma)| \neq \|\beta\||(\alpha,\gamma)|$, the vectors α and β are linearly independent.

As a result, the equality (6) holds if and only if the vector γ is a linear combination of the vectors α and β.

Expanding this determinant and using the condition $||\gamma|| = 1$, we obtain from (6) that

$$||\alpha||^2||\beta||^2 - (\alpha,\beta)^2 - F(x,y) \geq 0,$$

where $F(x,y)$ is a binary function defined by (2). And then using the inequality (4), the result follows at once.

Actually, the inequality (5) is a refinement of Cauchy-Schwarz's inequality.

3. Theorem and its corollaries

Now let us come to our main results.

Theorem *If* $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^2 < +\infty$, *then*

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 < (\sum_{n=1}^{\infty}\omega(n)a_n^2)(\sum_{n=1}^{\infty}\omega(n)b_n^2) - R_\omega^2, \tag{7}$$

where $\omega(n) = \pi - \frac{\theta(n)}{\sqrt{n}}$, $\theta(n) > 0$, *and* $R_\omega^2 > 0$.

Proof Define two double-sequences by

$$\alpha_{mn} = \frac{a_m}{(m+n)^{1/2}}(\frac{m}{n})^{1/4}, \quad \beta_{nm} = \frac{b_n}{(m+n)^{1/2}}(\frac{n}{m})^{1/4}, \tag{8}$$

and let $\alpha = (\alpha_{mn}), \beta = (\beta_{nm})$. An inner product of the vector α and β is defined by

$$(\alpha,\beta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}\beta_{nm}. \tag{9}$$

Applying the inequality (5) to estimate the right-hand side of (9) we obtain

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 = (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}\beta_{nm})^2 \leq (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}^2)(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{nm}^2) - R^2, \tag{10}$$

where R is defined by (3).

Notice that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}^2 = \sum_{n=1}^{\infty}\omega(n)a_n^2, \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{nm}^2 = \sum_{n=1}^{\infty}\omega(n)b_n^2,$$

where $\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m+n}(\frac{n}{m})^{1/2}$. According to the paper [2] we have

$$\omega(n) = \pi - \frac{\theta(n)}{\sqrt{n}},$$

where $\theta(n) > 0$ for all $n \in N$.

The norm of the vector x with weight $\omega(n)$ is denoted by

$$\|x\|_\omega = (\sum_{n=1}^{\infty}(\pi - \frac{\theta(n)}{\sqrt{n}})x_n^2)^{1/2}. \tag{11}$$

We write R in (10) in the form

$$R_\omega = \|a\|_\omega|(\beta,\gamma)| - \|b\|_\omega|(\alpha,\gamma)|, \tag{12}$$

where α and β are given by (8), and γ is chosen by

$$\gamma = \frac{6}{mn\pi^2}. \tag{13}$$

It is easy to deduce that $\|\gamma\| = (\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\frac{6}{mn\pi^2})^2)^{1/2} = (\frac{36}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1}^{\infty}\frac{1}{n^2})^{1/2} = 1$. Hence the vector γ satisfies the condition of Lemma 2, and it is obvious that $\|a\|_\omega|(\beta,\gamma)| \neq \|b\|_\omega|(\alpha,\gamma)|$. Whence the inequality (10) can be written as

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 \leq \|a\|_\omega^2\|b\|_\omega^2 - R_\omega^2, \tag{14}$$

where R_ω is given by (12).

Evidently, the vectors α, β and γ defined respectively by (8) and (13) are linearly independent, it is impossible to take the equality in (14). And owing to the fact that the conditions of Lemma 1 are satisfied, we have $R_\omega^2 > 0$.

Thus the proof of the theorem is completed.

Note According to the paper [2], the expression of $\theta(n)$ contained in (7) is given as follows:

$$\theta(n) = A_2(n) + C_2(n) - \frac{n}{2(n+1)},$$

where $A_2(n) = 2\sqrt{n}\arctan\frac{1}{\sqrt{n}}$ and $C_2(n) = -\sum_{k=1}^{s-1}\frac{\sqrt{n}B_{2k}}{(2k)!}(\frac{1}{n})^{2k}f^{(2k-1)}(\frac{1}{n}) + \rho_s$, $f(t) = \frac{1}{1+t}(\frac{1}{t})^{1/2}, t \in (0,1]$, $B_{j's}$ are Bernoulli numbers, viz. $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, etc., ρ_s is the remainder having the same sign as that of the first to be dropped, and having smaller absolute value in comparison.

Corollary 1 *If* $0 < \sum_{n=1}^{\infty} a_n^2 < +\infty$, *then*

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 < (\sum_{n=1}^{\infty}(\pi - \frac{\theta(n)}{\sqrt{n}})a_n^2)^2 - \tilde{R}_\omega^2,$$

where $\theta(n) > 0$ and $\tilde{R}_\omega^2 > 0$.

Proof We need only to show that $\tilde{R}_\omega^2 > 0$. We obtain from (12) that

$$\tilde{R}_\omega^2 = \|a\|_\omega^2(|(\beta,\gamma)| - |(\alpha,\gamma)|)^2, \tag{15}$$

where α and γ are given respectively by (8) and (13), and β_{nm} is defined by

$$\beta_{nm} = (\frac{a_n}{(m+n)^{1/2}}(\frac{n}{m})^{1/4}).$$

Notice that $\beta_{nm}/\alpha_{mn} = (\sqrt{n}a_n)/(\sqrt{m}a_m) \neq k$, for all $m, n \in N$, where k is a constant. Hence $\tilde{R}_\omega^2$ in (15) is not equal to zero. In other words, we have $\tilde{R}_\omega^2 > 0$.

Consequently, the corollary is proved.

Corollary 2 *With the assumptions as the theorem, we have*

$$(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n})^2 < (\sum_{n=1}^{\infty}(\pi - \frac{\xi}{\sqrt{n}})a_n^2)(\sum_{n=1}^{\infty}(\pi - \frac{\xi}{\sqrt{n}})b_n^2) - R_\omega^2, \tag{16}$$

where $\xi = \frac{\pi}{2} - \frac{7}{24} + \frac{\zeta}{320}(0 < \zeta < 1)$ *and* R_ω *is given by (12).*

Proof In the paper [3], it was shown that $\omega(n) \leq \pi - \frac{\xi}{\sqrt{n}}$, where $\omega(n)$ is a weight function of the form $\omega(n) = \sum_{m=1}^{\infty}\frac{1}{m+n}(\frac{n}{m})^{1/2}$ and $\xi = \frac{\pi}{2} - \frac{7}{24} + \frac{\zeta}{320}(0 < \zeta < 1)$.

Thus the corollary follows.

If R_ω^2 contained in (16) is replaced by zero, then the result of the paper [3] is yield. Clearly, the inequality (16) is an improvement of the corresponding result of the paper [3].

Similarly, some results in [5] and [6] might be improved.

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加权 Hilbert 不等式的一个改进

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摘 要: 本文通过引入一个适当的非零实数 R_ω 得到了加权 Hilbert 不等式的一个改进，并且利用 Gram 矩阵的正定性给出了 R_ω 的具体表达式.

关键词: 重级数；二次型；加权 Hilbert 不等式；权函数.