Journal of Mathematical Research & Exposition
Vol.24, No.2, 214-218, May, 2004

On Pronormal Minimal Subgroups of Finite Groups *

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Abstract: By using pronormal minimal subgroups and weak left Engel elements of prime order of the normalizers of Sylow subgroups of a finite group G, we obtain some sufficient conditions for G to be p-nilpotent, nilpotent and supersolvable respectively, which generalize some known results.

Key words: minimal subgroup; weak left Engel element; p-nilpotent; supersolvable.

Classification: AMS(2000) 20D10/CLC number: O152

Document code: A **Article ID:** 1000-341X(2004)02-0214-05

1. Introduction

Throughout this paper, G is a finite group, p and q are two distinct primes, $\pi(n)$ is the set of all distinct prime divisors of a positive integer n. G_p and $G_{p'}$ are a Sylow p-subgroup and a Hall p'-subgroup of G respectively. $H \propto K$ is the semidirect product of K by H. H sn G denotes that H is subnormal in G. The other notations are standard, mainly taken from [5] and [7].

A minimal subgroup of G is a subgroup of G of prime order. For a group of even order, it is also helpful to consider the cyclic subgroups of order 4. Itô [1,P.435] showed that if, for an odd prime p, every subgroup of G of order p lies in $Z(G)$, then G is p-nilpotent; and if all elements of G of order 2 and 4 lie in $Z(G)$, then G is 2-nilpotent. Buckley[2] showed that if all minimal subgroups of an odd order group are normal, then the group is supersolvable. These results have been extended by several authors (e.g., see [1] and [3]). Li Shirong[6] obtained "localized" and more general versions of the above conditions. Our main object is to generalize the main results of [6].

A subgroup H of G is called pronormal in G if H is conjugate to H^g in $\langle H, H^g\rangle$ for all $g \in G$. It is clear that H is normal in G if and only if it is both subnormal and pronormal in G.

An element x of G is called a weak left Engel element of G if there exists a positive integer n such that $[x, {}_n y] \in O_{\pi'}(G)$ for all elements y of G of prime power order with

*Received date: 2001-09-21
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$(|x|, |y|) = 1$, where $\pi = \pi(|x|)$. From this definition, the following facts are evident. If $x \in H \leq G$ and x is a weak left Engel element of G, then x is a weak left Engel element of H. If N is normal in G and x is a weak left Engel element of G such that $(|x|, |N|) = 1$, then xN is a weak left Engel element of G/N.

2. Preliminaries

We would like to cite the following results which we will use in the proof of our main results.

Lemma 1 *Suppose that $p \in \pi(G)$ and N is normal in G. Then every cyclic subgroup P_0 of N_p of order p or 4 (when $p = 2$) is pronormal in $N_G(N_p)$ if and only if P_0 is pronormal in G.*

Proof Only the necessity of the condition is in doubt. By hypothesis, P_0 is pronormal in $N_G(N_P)$. And P_o sn $N_G(N_p)$. Hence P_0 is normal in $N_G(N_P)$ and so P_0^g is normal in $N_G(N_p^g)$ for all $g \in G$, that is, every cyclic subgroup P_0^g of N of order p or 4 (if $p = 2$) is normal in the normalizers of Sylow p-subgroups of N in G which contain P_0^g. Put $H = \langle P_0, P_0^x\rangle$ for $x \in G$. Let P_1 be a Sylow p-subgroup of H containing P_0. By hypothesis $P_0^x \leq P_1^y$ for some $y \in H$, that is, $P_0^{xy^{-1}} \leq P_1$. Note that if a Sylow p-subgroup G_p of G contains N_p then $G_p \leq N_G(N_P)$. From this we may assume that $P_1 \leq G_p \leq N_G(N_p)$ and so P_0 is normal in G_p. Since $P_0^{xy^{-1}}$ is a subgroup of N of order p or 4, $P_0^{xy^{-1}}$ is normal in G_p. Thus G_p and $G_p^{yx^{-1}}$ are conjugate in $N_G(P_0)$, that is, $G_p{=}G_p^{yx^{-1}n}$, where $n \in N_G(P_0)$. Then $yx^{-1}n \in N_G(G_p)$. Since $N_p = N \bigcap G_p$, $N_G(G_p) \leq N_G(N_p)$ and so $yx^{-1}n \in N_G(P_0)$, that is, $P_0^{yx^{-1}n} = P_0$. Hence $P_0^y = P_0^x$. This means that P_0 is pronormal in G.

Lemma 2 *Let G be a minimal non-p-nilpotent group. Then:(1) $G = G_q \propto G_p$, G_q is not normal in G and $G_q = \langle a\rangle$ for some $a \in G$; (2) G_p has exponent p if $p > 2$ and exponent at most 4 if $p = 2$; (3) If $x \in G_p \setminus \Phi(G_p)$ then $[x, a] \in G_p \setminus \Phi(G_p)$.*

Proof (1) and (2) see [5,P.434].

(3) Let $x \in G_p \setminus \Phi(G_p)$. Then $[x, a] \in G_p$ since G_p is normal in G. We will prove that $[x, a] \notin \Phi(G_p)$ in two cases below.

Case (i): G_p is cyclic. Then $G_p = \langle x\rangle$. If $p = \min\pi(G)$, then, by [7,10.1.9], G is p-nilpotent, a contradiction. Hence $p \neq \min\pi(G)$, then $|x| = p$ and $\Phi(G_p) = 1$. If $[x, a] = 1$ then $G = G_q \times G_p$, contradicting that G_q is not normal in G. Hence $[x, a] \neq 1$, so $[x, a] \notin \Phi(G_p)$.

Case (ii): G_p is not cyclic. Conjugation induces a representation of G_q by linear transformations of the vector space $V = G_p/\Phi(G_p)$. Suppose that $[x, a] \in \Phi(G_p)$. Then $\langle x\Phi(G_p)\rangle$ is a G_q-invariant subspace of V, so, by Maschke's theorem, there exists a G_q-invariant subspace $U = H/\Phi(G_p)$ such that $V = \langle x\Phi(G_p)\rangle \oplus U$. But G_p is not cyclic, so HG_q and $\langle x\rangle\Phi(G_p)G_q$ are proper subgroups of G, so are p-nilpotent. In particular H, $\langle x\rangle\Phi(G_p) \leq N_G(G_q)$, so G_q is normal in G, a contradiction. Hence $[x, a] \notin \Phi(G_p)$.

Note By Satz III.5.2 of [5], a minimal nonilpotent group is a minimal non-p-nilpotent group.

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Lemma 3 *(1) If $p = \min\pi(G)$ and every cyclic subgroup of G_p of order p or 4 (when $p = 2$) is pronormal in G, then G is p-nilpotent. (2) If $p \neq min\pi(G)$ and, for each element x of G_p of order p, x is a weak left Engel element of G, then G is p-nilponent.*

Proof Assume false and let G be a counterexample of minimal order. It is easy to check that every proper subgroup of G either is p-nilpotent or inherits the hypothesis and so G is a minimal non-p-nilpotent group. By Lemma 2, $G = G_q \propto G_p$, $G_q = \langle a\rangle$ for some $a \in G$, G_p has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Let $x \in G_p \setminus \Phi(G_p)$. We discuss in two cases below.

Case: $p = \min\pi(G)$. Then $|x|$ is p or 4 (possibly $p = 2$). By hypothesis, $\langle x\rangle$ is pronormal in G. And $\langle x\rangle$ sn G. Thus $\langle x\rangle$ is normal in G and so $\langle x\rangle G_q = G_q\langle x\rangle$. If $G = G_q\langle x\rangle$, by [7,10.1.9], G is p-nilpotent, a contradiction. Thus $G_q\langle x\rangle < G$ and so $G_q\langle x\rangle = G_q \times \langle x\rangle$. This means that $[x, a] = 1$. By Lemma 2, $[x, a] \in G_p \setminus \Phi(G_p)$, a contradiction.

Case: $p \neq \min\pi(G)$. Then $|x| = p$ and x is a weak left Engel element of G by hypothesis. Thus there exists a positive integer n such that $[x, {}_na] \in O_{p'}(G) \bigcap G_p = 1$. By lemma 2 again, $[x, a] \in G_p \setminus \Phi(G_p)$. Repeating this argument yields that $[x, {}_na] \neq 1$. This contradiction completes the proof.

2. Main results

Theorem 1 *Let N be a normal subgroup of G such that G/N is p-nilpotent. Then: (1) If $p = min\pi(G)$ and every cyclic subgroup of N_p of order p or 4 (when $p = 2$) is pronormal in $N_G(N_p)$, then G is p-nilpotent. (2) If $p \neq min\pi(G)$ and every element x of N_p of order p is a weak left Engel element of $N_G(N_p)$ and $\langle x\rangle$ is pronormal in N_p, then G is p-nilpotent.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. It is easy to check that (N, N) satisfies the hypothesis for (G, N). If $N \neq G$ then N is p-nilpotent by the minimality of G. So $N_{p'}$ is normal in G. It is clear that $(G/N_{p'}, N/N_{p'})$ satisfies the hypothesis for (G, N). If $N_{p'} \neq 1$ then $G/N_{p'}$ is p-nilpotent and so G is p-nilpotent, a contradiction. It forces that $N_{p'}=1$ so that $N=N_p$. By hypothesis G/N is p-nilpotent. Let M be the inverse image of the normal p-complement of G/N in G. Evidently (M, N) satisfies the hypothesis for (G, N). If $M \neq G$ then M is p-nilpotent as G is a minimal counterexample. Note that the normal p-complement of M is also the normal p-complement of G. Hence G is p-nilpotent. From this contradiction, we deduce that $N = G$ and $N_p = G_p$.

Assume that $p = \min\pi(G)$. By hypothesis every cyclic subgroup of G_p of order p or 4 (if $p = 2$) is pronormal in $N_G(G_p)$. It follows from Lemma 1 that every cyclic subgroup of G_p of order p or 4 is pronormal in G. Thus G is p-nilpotent by Lemma 3, a contradiction. Hence G is p-nilpotent and the claim (1) follows.

Now assume that $p \neq \min\pi(G)$. By Lemma 3, $N_G(G_p)$ is p-nilpotent. Since G is not p-nilpotent, we may assume that H is a minimal non-p-nilpotent subgroup of G. By Lemma 2, $H = H_q \propto H_p$ and H_p as exponent p. We may without loss of generality assume that $H_p \leq G_p$. By hypothesis, every minimal subgroup of G_p is pronormal in G_p and so normal in G_p. This implies that $H_p \leq Z(G_p)$. So $G_p \leq C_G(H_p)$ is normal in $N_G(H_p)$.

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Put $K = N_G(H_p)$. By the Frattini argument, $K = C_G(H_p)N_K(G_p)$. Since $N_G(G_p)$ is p-nilpotent and so is $N_K(G_p)$, thus $N_K(G_p) = G_p \times C$, where C is the normal p-complement of $N_K(G_p)$. Note that $H_p \leq Z(G_p)$, hence $G_p \times C \leq C_G(H_p) \leq K$ and so $K = C_G(H_p)$. But $H_q \leq N_H(H_p) \leq K$. Thus $H_q \leq C_G(H_p)$, that is, $H = H_q \times H_p$, contradicting that H_q is not normal in H. Therefore G is p-nilpotent and the claim (2) is true.

Theorem 2 *Let N be a normal subgroup of G such that G/N is nilpotent. If, for each $p \in \pi(N)$ and each element x of N_p of order p or 4(possibly $p = 2$), when $p = \min \pi(G)$, $\langle x \rangle$ is pronormal in $N_G(N_p)$, and when $p \neq \min \pi(G)$, x is a weak left Engel element of $N_G(N_p)$, then G is nilpotent.*

Proof Assume false and let G be a counterexample of minimal order. If $N = G$, then $N_p = G_p$ for each $p \in \pi(G)$. Let $q = \min \pi(G)$. By Lemma 1, every cyclic subgroup of G_q with order q or 4(if $q = 2$) is pronormal in G. By Lemma 3, G is q-nilpotent and so $G_{q'}$ is the normal q-complement of G. Since G is not nilpotent, G has a minimal nonnilpotent subgroup, K say. Thus K is q-nilpotent. By Lemma 2, $K = K_q \propto K_p$, $K_q = \langle a \rangle$ for some $a \in K$ and K_p has exponent p. If G is a $\{p,q\}$-group, then G_p is normal in G as G is q-nilpotent and so $K_p \leq G_p$. Let $x \in K_p \setminus \Phi(K_p)$. Then x is a weak left Engle element of $G = N_G(G_p)$. Hence there exists a positive integer n such that $[x,_n a] \in O_{p'}(K) \bigcap K_p = 1$. By Lemma 2, $[x,a] \in K_p \setminus \Phi(K_p)$ and so $[s,a] \neq 1$. Repeating this argument yields that $[x,_n a] \neq 1$, a contradiction. Hence $|\pi(G)| \geq 3$. Since G is q-nilpotent with $q = \min \pi(G)$, G is solvable by Feit-Thompson theorem. Then G contains a Hall $\{q,p\}$-subgroup G_qG_p such that $K \leq G_qG_p$. It is easy to check that G_qG_p satisfies the conditions of Theorem 2 and so G_qG_p is nilpotent by the minimality of G, contradicting that K is not nilpotent. This forces that $N < G$.

Repeating the argument in the precedent paragraph, we can show that N is nilpotent. Since G is not nilpotent, G contains a minimal nonnilpotent subgroup, H, say. By Lemma 2, $H = H_q \propto H_p$, $H_q = \langle a \rangle$ for some $a \in H$ and H_p has exponent p or 4(if $p = 2$). $H/H \bigcap N$ is nilpotent as $H/H \bigcap N \simeq HN/N \leq G/N$. But H_q is not normal in H. This forces that $H_p \leq N$. By Sylow's theorem we may assume that $H_p \leq N_p$. By the Frattini argument, $G = NN_G(N_p)$. Note that $H \nleq N$. This forces that $H_q \leq N_G(N_p)$. Let $x \in H_p \setminus \Phi(H_p)$.

If $p = \min \pi(G)$, then $|x| = p$ or 4 (possibly $p = 2$). By hypothesis, $\langle x \rangle$ is pronormal in $N_G(N_p)$. But $\langle x \rangle$ sn $N_G(N_p)$. Thus $\langle x \rangle$ is normal in $N_G(N_p)$ and so $\langle x \rangle H_q = H_q \langle x \rangle$. By [7,10.1.9], $H_q\langle x \rangle$ is p-nilpotent and so nilpotent. This holds for each generator of H_p. Hence H is nilpotent, a contradiction.

So we may assume that $p \neq \min \pi(G)$. Then $|x| = p$. By hypothesis, x is a weak left Engel element of $N_G(N_p)$. Repeating the argument in the first paragraph we can derive a final contradiction. The theorem follows.

Theorem 3 *Let N be a normal subgroup of G such that G/N is supersolvable. If every cyclic subgroup of N_p of order p or 4 (when $p = 2$) is pronormal in $N_G(N_p)$ for each $p \in \pi(N)$, then G is supersolvable.*

Proof For each $p \in \pi(N)$, by hypothesis, every cyclic subgroup K of N_p with order p or 4(if p=2) is pronormal in $N_G(N_p)$. Moreover, K sn $N_G(N_p)$. Hence K is normal in

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$N_G(N_p)$. Recall that N is normal in G. Now we see that all G-chief factors which are contained in N are cyclic by [8, Theorem IX.6.8]. Therefore, G is supersolvable.

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关于有限群的 Pronormal 极小子群

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摘 要: 利用有限群 G 的 pronormal 极小子群和 Sylow 子群正规化子中的素数阶弱左 Engel 元素得到了 G 成为 p- 幂零群、幂零群和超可解群的一些充分条件，这些结果推广了已知结论.

关键词: 极小子群；弱左 Engel 元素； p- 幂零；超可解.

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