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The Irreducibility and Isomorphism of Piecewise Algebraic Sets *

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Abstract: In this paper, we define the piecewise algebraic sets by using multivariate spline functions and discuss their irreducibility and isomorphism problem. We present two equivalent conditions for the irreducibility of piecewise algebraic sets, and turn the isomorphism and classifying problem of piecewise algebraic sets into the isomorphism and classifying problem of commutative algebras.

Key words: multivariate spline function; piecewise algebraic sets; irreducibility; isomorphism.

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1. Introduction

Piecewise algebraic variety is a generalization and development of the algebraic variety. R. H. Wang[2] introduced the piecewise algebraic curve for studying the suitability of interpolation nodes of bivariate spline function many years ago.

Algebraic geometry deals with the solutions of systems of polynomial equations in an affine or projective space. In other words, it studies the algebraic varieties. By using the multivariate splines to replace the polynomials defined in the algebraic variety, we can define the piecewise algebraic set. Some basic results on the piecewise algebraic varieties have been given in [2,3]. In this paper, we study the irreducibility and isomorphism problems of piecewise algebraic sets.

2. The irreducibility of piecewise algebraic sets

Let D be a domain in k^n, k a fixed algebraically closed field of characteristic zero, and Δ a partition of D consists of a finite number of hyperplanes

$$l_i(x_1,\dots,x_n) = a_{i1}x_1 + \cdots + a_{in}x_n + b_i = 0, \quad i = 1,\dots,N.$$

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Denote by $S^\mu(\Delta)$ the C^μ spline function ring, a k-algebra. In fact, $S^\mu(\Delta)$ is a $k[x_1,\dots,x_n]$-algebra. By [6], any multivariate spline function s in $S^\mu(\Delta)$ has the following representation

$$s(x_1,\dots,x_n) = p(x_1,\dots,x_n) + \sum [l_i(x_1,\dots,x_n)]_*^{\mu+1} q_i(x_1,\dots,x_n),$$

where $q_i(x_1,\dots,x_n)$ are the smoothing cofactors satisfying the conformality condition, so that $S^\mu(\Delta)$ is a finitely generated $k[x_1,\dots,x_n]$-module that generated by its elements

$$1, [l_1(x_1,\dots,x_n)]_*^{\mu+1}, \dots, [l_m(x_1,\dots,x_n)]_*^{\mu+1}.$$

Since $k[x_1,\dots,x_n]$ is a Noetherian ring, we have

Proposition 2.1 *$S^\mu(\Delta)$ is a reduced Noetherian ring, but not an integral domain.*

As a generalization of the algebraic set, we define piecewise algebraic set by using the multivariate spline functions.

Definition 2.2 *Let Δ be a partition of k^n. A subset Y of k^n is called a piecewise algebraic set respect to partition Δ, if there exists a subset T of $S^\mu(\Delta)$ such that*

$$Y = Z(T) = \{P \in k^n | s(P) = 0, \quad \text{for all} \quad s \in T\}.$$

It is clear that if I is an ideal of $S^\mu(\Delta)$ generated by T, then

$$Z(T) = Z(I).$$

Furthermore, since $S^\mu(\Delta)$ is a Noetherian ring, any ideal has a finite set of generators $s_1,\dots,s_t$, $Z(T)$ can be expressed as the common zeros of the finite set of splines $s_1,\dots,s_t$.

If $s \in S^\mu(\Delta)$, we obtain a piecewise algebraic set $Y = Z(s)$, which is called a piecewise algebraic curve if $n = 2$, or a piecewise algebraic surface if $n = 3$, or a piecewise algebraic hypersurface if $n \geq 3$.

Definition 2.3 *A piecewise algebraic set Y is called to be irreducible if $Y = Y_1 \cup Y_2$ in which Y_1 and Y_2 are piecewise algebraic sets, implies $Y = Y_1$ or $Y = Y_2$. An irreducible piecewise algebraic set is called a piecewise algebraic variety.*

Proposition 2.4[2] *Every piecewise algebraic set can be expressed uniquely as a union of piecewise algebraic varieties, no one containing another.*

Next, we need to explore the relationship between subsets of k^n and ideals in $S^\mu(\Delta)$ more deeply. For any subset Y of k^n, the ideal of Y in $S^\mu(\Delta)$ is defined by

$$I(Y) = \{s \in S^\mu(\Delta) | s(P) = 0 \quad \text{for all} \quad P \in Y\}.$$

Now there are two functions Z and I, which map subsets of $S^\mu(\Delta)$ to piecewise algebraic sets and subsets of k^n to ideals of $S^\mu(\Delta)$ respectively. Their properties are summarized in the following proposition.

Proposition 2.5 *If S and T are subsets of $S^\mu(\Delta)$, X and Y are subsets of k^n, then*

(1) If $S \subseteq T$, then $Z(S) \supseteq Z(T)$.
(2) If $X \subseteq Y$, then $I(X) \supseteq I(Y)$.
(3) $I(X \cup Y) = I(X) \cap I(Y)$.
(4) $ZI(X) = \overline{X}$, the closure of X.
(5) $S \subseteq IZ(S)$; $X \subseteq ZI(X)$.
(6) $Z(S) = ZIZ(S)$; $I(X) = IZI(X)$.

Proposition 2.6 *A piecewise algebraic set Y is irreducible if and only if its ideal $I(Y)$ is a prime ideal of $S^\mu(\Delta)$.*

Proof if Y is irreducible, we show that $I(Y)$ is prime. Indeed, if $s \cdot t \in I(Y)$, then

$$Y \subseteq Z(st) = Z(s) \cup Z(t).$$

Thus

$$Y = (Y \cap Z(s)) \cup (Y \cap Z(t)).$$

Since Y is irreducible, we have either $Y = Y \cap Z(s)$, in which case $Y \subseteq Z(s)$, or $Y \subseteq Z(t)$. Hence either $s \in I(Y)$ or $t \in I(Y)$.

Conversely, if $P = I(Y)$ is a prime ideal, and suppose that $Y = Y_1 \cup Y_2$, then

$$P = I(Y_1) \cap I(Y_2),$$

so either $P = I(Y_1)$, or $P = I(Y_2)$. Thus

$$Y = Y_1 \text{ or } Y = Y_2,$$

and Y is irreducible. □

In this section, we will give an equivalent proposition for a irreducible piecewise algebraic set.

Definition 2.7 *If $Y \subseteq k^n$ is a piecewise algebraic set, we define the coordinate ring $S(Y)$ of Y to be $S^\mu(\Delta)/I(Y)$. If Y is a piecewise algebraic variety, then $S(Y)$ is an integral domain.*

Definition 2.8 *Let $\overline{k}$, k' be two extended fields of k, $\xi = (\xi_1, \ldots, \xi_n) \in \overline{k}^n$, $\eta = (\eta_1, \ldots, \eta_n) \in k'^n$, if for arbitrary element s of $S^\mu(\Delta)$, $s(\xi) = 0$ implies $s(\eta) = 0$, then we call the ξ to be a generic point of η with respect to k or η to be a k-specialization of ξ. Let Y be a piecewise algebraic set, If $I(Y)$ has an extension zero ξ such that all extension zeros of $I(Y)$ are k-specialization of ξ, then ξ is called a generic point of Y.*

Theorem 2.9 *A piecewise algebraic set Y is irreducible if and only if Y has a generic point.*

Proof If Y has a generic point ξ, we show that $I(Y)$ is a prime ideal of $S^\mu(\Delta)$.

Indeed, if $fg \in I(Y)$, then $f(\xi)g(\xi) = 0$, thus either $f(\xi) = 0$, or $g(\xi) = 0$. Assume $f(\xi) = 0$, since ξ is a generic point of Y, then for very extension zero η of $I(Y)$ we have $f(\eta) = 0$, thus $f \in I(Y)$. In the same way, we can prove that $g(\xi) = 0$ implies $g \in I(Y)$. So $I(Y)$ is a prime ideal. Hence by Proposition 2.6, we know that Y is irreducible.

Conversely, if Y is irreducible, by Proposition 2.6, $I(Y)$ is prime ideal of $S^\mu(\Delta)$. Hence the coordinate ring $S(Y)$ of Y is an integral domain, the quotient field $\overline{k} = K(Y)$ of $S(Y)$ is an extended field of k. Denote by ξ_i the homomorphic image of x_i in k. Thus $\xi = (\xi_1, \ldots, \xi_n) \in \overline{k}^n$. Next we will prove that ξ is a generic point of Y.

First we prove that ξ is an extension zero of $I(Y)$. For arbitrary $f(x_1, \ldots, x_n) \in I(Y)$, the homomorphic image $\underline{f \text{ in } S(Y)}$ is zero, and hence the homomorphic image of f in $\overline{k} = k(Y)$ is also zero, i.e., $\overline{f(x_1, \ldots, x_n)} = \overline{0}$. Thus $f(\xi_1, \ldots, \xi_n) = \overline{0}$, so ξ is an extension zero of $I(Y)$.

Assume that $f(x_1, \ldots, x_n) \in S^\mu(\Delta)$ of satisfying $f(\xi_1, \ldots, \xi_n) = \overline{0}$. Thus

$$\overline{f(x_1, \ldots, x_n)} = \overline{0}, \quad f \in I(Y),$$

and for every extension zero η of $I(Y)$, $f(\eta) = 0$, namely, $f(\xi) = 0$ implies $f(\eta) = 0$. So all extension zeros of $I(Y)$ are k-specialization of ξ, with ξ being a generic point of Y.

3. The isomorphism of piecewise algebraic sets

In section one, we attached to each piecewise algebraic set Y a coordinate ring $S(Y)$. We ask if isomorphic coordinate rings determine the "same" piecewise algebraic sets in some sense. For solving the problem, we will give the following definition and result.

Definition 3.1 *Let $V \subseteq k^n$ be a piecewise algebraic set respect to partition Δ, $W \subseteq k^m$ be a piecewise algebraic set respect to partition Δ'. A mapping $f : V \to W$ is called a spline mapping if there are splines $f_i(x_1, \ldots, x_n) \in S^\mu(\Delta)$ $(1 \le i \le m)$ such that*

$$f = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

If $f : V \to W$ and $g : W \to V$ are spline mappings such that

$$f \circ g = \mathbf{1}|_W, \ g \circ f = \mathbf{1}|_V,$$

then we say V and W are isomorphic.

Theorem 3.2 *Let $V \subseteq k^n$ be a piecewise algebraic set of Δ, $W \subseteq k^m$ be a piecewise algebraic set of Δ', $\mathrm{Hom}(V, W)$ denote the set of all spline mappings from V to W, $\mathrm{Hom}(S(W), S(V))$ denotes the set of all k-algebraic homomorphisms from $S(W)$ to $S(V)$. Then the mapping $\mathrm{Hom}(V, W) \to \mathrm{Hom}(S(W), S(V))$*

$$f \longmapsto f^* : s \mapsto s \circ f$$

is surjective. Moreover $f : V \to W$ is isomorphic if and only if $f^* : S(W) \to S(V)$ is k-isomorphic.

Proof Let $f \in \mathrm{Hom}(V, W)$. Then for every $g \in S(W)$, $g : W \to k$, we have a spline mapping $g \circ f : V \to k$. Thus f induces a mapping:

$$f^* : S(W) \to S(V), \quad f^* = g \circ f.$$

Assume that V and W are not empty, $S(W)$, $S(V)$ are all k-algebra and contain k as their subalgebra. Every element $a \in k$ is regarded as an element of $S(W)$, thus a is a spline function on W which maps all elements of W to a. Obviously $f^*(a) = a$. This shows $f^*|_k = \mathbf{1}$. It is easy to prove

$$f^*(g_1 + g_2) = f^*(g_1) + f^*(g_2), \quad f^*(g_1 g_2) = f^*(g_1) f^*(g_2).$$

So $f^* : S(W) \to S(V)$ is a k-algebraic homomorphism, and it has the following properties

(1) If $g : W \to U$ is a spline mapping, then $(g \circ f)^* = f^* g^* : S(U) \to S(V)$.

(2) $(\mathbf{1}|_V)^* = \mathbf{1}_{S(V)}$.

Thus if V and W are isomorphic, then there exist spline function $f : V \to W$ and $g : W \to V$ such that $f \circ g = \mathbf{1}|_W$, $g \circ f = \mathbf{1}|_V$, hence by properties (1) and (2), we know

$$f^* g^* = (g \circ f)^* = (\mathbf{1}|_V)^* = \mathbf{1}_{S(V)},$$
$$g^* f^* = (f \circ g)^* = (\mathbf{1}|_W)^* = \mathbf{1}_{S(W)}.$$

So $f^* : S(W) \to S(V)$ is a k-algebraic isomorphism.

Conversely, let $\varphi : S(W) \to S(V)$ be a k-algebraic homomorphism. Consider special spline functions $y_i \in S(W)$. Then φ gives spline functions

$$\varphi(y_i) = f_i(x_i, \dots, x_n) \in S(V), \; i = 1, \dots, m.$$

Thus we have

$$f = (f_1, \dots, f_m) : k^n \longrightarrow k^m,$$
$$(a_1, \dots, a_n) \longmapsto (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)).$$

We prove first that $f(V) \subseteq W$. For every $g(y_1, \dots, y_m) \in I(W)$, $g(y_1, \dots, y_m)$ is zero element of $S(W)$. Since φ is a k-algebra homomorphism, it maps zero to zero, hence

$$\varphi(g(y_1, \dots, y_n)) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

is the zero element of $S(V)$, thus $\varphi(g) \in I(V)$. It shows that if $(a_1, \dots, a_n) \in V$, then for every $g(y_1, \dots, y_m) \in I(W)$,

$$g(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) = 0,$$

hence

$$g(f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)) \in W.$$

Thus $f(V) \subseteq W$, so $f : V \to W$ is a spline mapping.

Second, we prove that $f^* = \varphi$. Since for every $g(y_1, \dots, y_m) \in I(W)$,

$$f^*(g(y_1, \dots, y_m)) = g(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) = \varphi(g(y_1, \dots, y_n)),$$

hence $f^* = \varphi$. Thus we have the following result: for every k-algebraic homomorphism $\varphi : S(W) \to S(V)$, there exists a spline mapping $f : V \to W$ such that $f^* = \varphi$.

Finally, if φ is the identity homomorphism of $S(W)$, then the f obtained as above is an identity mapping on W, hence if $\varphi : S(W) \to S(V)$ is k-algebraic isomorphism, then $f : V \to W$ is an isomorphism of piecewise algebraic sets. □

We have turned the isomorphism and classification problem of piecewise algebraic sets into the pure algebraic problem of isomorphism and classificationof k-commutative algebras.

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分片代数集合的不可约性和同构

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摘 要: 本文利用多元样条函数来定义分片代数集合，讨论了分片代数集合的不可约性和同构问题，给出了分片代数集合不可约的两个等价条件，并把分片代数集合的同构分类问题转化为交换代数的同构分类问题.

关键词: 多元样条函数；分片代数集合；不可约性；同构.