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L^p Estimates for Riesz Transform Associated to Schrödinger Operator *

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Abstract: In this paper we consider the boundedness of Riesz transform associated to uniformly elliptic operators $L = -\text{div}(A(x)\nabla) + V(x)$ with non-negative potentials V on $\mathbf{R}^n$ which belonging to certain reverse Hölder class.

Key words: Riesz transform; Schrödinger operators.

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1. Introduction

For Schrödinger operators $-\Delta + V(x)$ with non-negative polynomials V, several authors (cf. Shen, Zhong etc.) studied the L^p boundedness for $1 < p < \infty$ of $\nabla(-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2}\nabla, \nabla(-\Delta + V)^{-1}\nabla, V^{1/2}\nabla(-\Delta + V)^{-1}$, and $\nabla^2(-\Delta + V)^{-1}$. In particular, Zhong[10] proved that if V is a non-negative polynomial, $\nabla^2(-\Delta + V)^{-1}, \nabla(-\Delta + V)^{-1/2}, \nabla(-\Delta + V)^{-1}\nabla$ are C-Z operators. It is well-known that C-Z operators are bounded on L^p, for $1 < p < \infty$. Shen[8,9] generalized these results. He proved that $\nabla(-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are C-Z operators if V belongs to reverse Hölder class B_n. Recently, Kurata and Sugano[8] considered uniformly elliptic operators $L = -\text{div}(A(x)\nabla) + V(x)$ with non-negative potentials V on $\mathbf{R}^n (n \geq 3)$ which belong to certain reverse Hölder class and gave several estimates for $VL^{-1}, V^{-1/2}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on weighted L^p spaces.

In this paper, we consider uniformly elliptic operators

$$L = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j) + V(x) := L_0 + V(x)$$

with certain non-negative potentials $V(x)$ on $\mathbf{R}^n (n \geq 3)$, where $a_{ij}(x)$ are measurable functions satisfying the conditions:

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(A1) There exists a constant $\lambda \in (0,1]$ such that

$$a_{ij}(x) = a_{ji}(x), \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \quad x,\xi \in \mathbf{R}^n;$$

(A2) There exist constants $\alpha \in (0,1]$ and $K > 0$ such that

$$\|a_{ij}\|_{C^\alpha(\mathbf{R}^n)} \le K.$$

(A3) $a_{ij}(x+z) = a_{ij}(x), \quad x \in \mathbf{R}^n, z \in \mathbf{Z}^n.$

Throughout this paper we use the following notation:

$$\partial_j = \nabla_j = \nabla_{x_j} = \frac{\partial}{\partial x_j}, \quad |\nabla u(x)|^2 = \sum_{j=1}^{n} |\nabla_j u(x)|^2.$$

When $A(x)$ satisfies (A1)-(A3), Alexopoulos got in [1] that $T_0 = \nabla L_0^{-1/2}$ is bounded on $L^p(1 < p < \infty)$ and weakly bounded on L^1. When L_0 associated with a complex matrix A satisfying uniform elliptic condition, in [3] it was proved that the operator T_0 is bounded from Hardy space $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$, hence by interpolation, is bounded on $L^p(1 < p \le 2)$ under the following assumptions:

(i) The analytic semigroup e^{-tL_0} generated by L_0 has kernels which posses Gaussian upper bounds and Hölder continuity bounds in their space variables;

(ii) $T_0 = \nabla L_0^{-1/2}$ is bounded on L^2.

Later Duong and McIntosh[4] considered the $L^p(\Omega)$ boundedness of T_0 without the assumption of Hölder continuity in the space variables of the kernels of the semigroup e^{-tL}, where Ω is a domain of $\mathbf{R}^n$ without any assumptions on the smoothness of its boundary.

The purpose of this paper is to show the boundedness of the operators $\nabla L^{-1/2}, V^{1/2}L^{-1/2}$ on L^p spaces. Actually the pointwise estimate of $V^{1/2}L^{-1/2}$ by the Hardy-Littlewood maximal function tells us more information about these operators. It extends the results in [9] to uniformly elliptic operators with non-negative potentials.

Definition 1 *Let $V(x) \ge 0$.*

(i) For $1 < q < \infty$, we say $V \in B_q$, if $V \in L^q_{\mathrm{loc}}(\mathbf{R}^n)$ and there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{1/q} \le \frac{C}{|B|}\int_B V(x)\mathrm{d}x \tag{1}$$

holds for every ball $B \in \mathbf{R}^n$.

(ii) We say $V(x) \in B_\infty$, if $V \in L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ and there exists a constant C such that

$$\|V\|_{L^\infty(B)} \le \frac{C}{|B|}\int_B V(x)\mathrm{d}x$$

holds for every ball $B \in \mathbf{R}^n$.

One remarkable feature about the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (1), such that $V \in B_{q+\varepsilon}$. For

$1 < q_1 < q_2 < \infty$, it is easy to see $B_\infty \subset B_{q_2} \subset B_{q_1}$. It's well-known that $V \in B_q$ implies $V \in A_\infty$ (Muckenhoupt weight class).

2. Main results

We give some fundamental properties of functions in the B_q class.

Proposition 2 *If $V(x) \in B_q (1 < q \leq \infty), \lambda$ is a non-negative constant, then $V(x) + \lambda \in B_q$.*

In fact, for all ball $B \in \mathbf{R}^n$, if $1 < q < \infty$

$$\begin{aligned}&(\frac{1}{|B|}\int_B (V(x)+\lambda)^q \mathrm{d}x)^{1/q}\\ &\leq (\frac{1}{|B|}\int_B V(x)^q \mathrm{d}x)^{1/q} + (\frac{1}{|B|}\int_B \lambda^q \mathrm{d}x)^{1/q}\\ &\leq C(\frac{1}{|B|}\int_B V(x)\mathrm{d}x + \lambda) = \frac{C}{|B|}\int_B (V(x)+\lambda)\mathrm{d}x.\end{aligned}$$

The case $q = \infty$ is similar.

Let $V(x) \in B_{n/2}$ and $V(x) \neq 0$. Then the function $m(x, V)$ is well-defined by

$$\frac{1}{m(x,V)} = \sup\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\mathrm{d}y \leq 1\}. \tag{2}$$

If $r_0 = m(x, V)^{-1}$, then $\frac{1}{{r_0}^{n-2}}\int_{B(x,r_0)} V(y)\mathrm{d}y = 1$. It is not difficult to find if $\lambda \geq 0$ is a constant, then $m(x, \lambda) = C\sqrt{\lambda}$.

Proposition 3 *For $V(x) \in B_{n/2}, \lambda \geq 0$, we have $m(x, V) \leq m(x, V + \lambda)$.*

In fact,

$$\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\mathrm{d}y \leq 1\} \supseteq \{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} (V(y)+\lambda)\mathrm{d}y \leq 1\}.$$

Lemma 4[8] *Let $k > 0$ be an integer.*

(i) Suppose (A1) for $A(x)$. Under the assumption $V(x) \in B_{n/2}$, there exists a constant C_k such that

$$0 \leq \Gamma(x,y) \leq \frac{C_k}{(1+m(x,V)|x-y|)^k |x-y|^{n-2}}.$$

(ii) Suppose (A1) and (A2) for $A(x)$. Under the assumption $V(x) \in B_n$, there exists a constant C_k such that

$$|\nabla_x \Gamma(x,y)| \leq \frac{C_k}{(1+m(x,V)|x-y|)^k |x-y|^{n-1}},$$

where $\Gamma(x, y)$ is the fundamental solution of L.

Let $\Gamma(x, y; \lambda), \Gamma_0(x, y; \lambda)$ be the fundamental solution of $L + \lambda$ and $L_0 + \lambda$, respectively. By Propositions 2 and 3, it is easily to find that Lemma 4 is still true when we replace

$\Gamma(x,y)$ with $\Gamma(x,y;\lambda)$ if $\lambda > 0$. More precisely, we have

Proposition 5 *Let $k > 0$ be an integer, $\lambda > 0$.*

(i) Suppose (A1) for $A(x)$. Under the assumption $V(x) \in B_{n/2}$, there exists a constant C_k such that

$$0 \leq \Gamma(x,y;\lambda) \leq \frac{C_k}{(1+m(x,V)|x-y|)^k(1+\lambda^{1/2}|x-y|)^k|x-y|^{n-2}}.$$

(ii) Suppose (A1) and (A2) for $A(x)$. Under the assumption $V(x) \in B_n$, there exists a constant C_k such that

$$|\nabla_x\Gamma(x,y;\lambda)| \leq \frac{C_k}{(1+m(x,V)|x-y|)^k(1+\lambda^{1/2}|x-y|)^k|x-y|^{n-1}}.$$

Lemma 6 *Suppose $V \in B_q, n/2 < q < n$, $(L+V+\lambda)u = 0$ in $B(x_0,2R)$. Then for $x \in B(x_0,R)$*

$$|\nabla u(x)| \leq \int_{B(x_0,2R)} \frac{V(y)|u(y)|}{|x-y|^{n-1}}\mathrm{d}y + \frac{C}{R^{n+1}}\int_{B(x_0,2R)}|u(y)|\mathrm{d}y.$$

Proof Let $\eta \in C_0^\infty(B(x_0,2R))$ such that $\eta = 1$ on $B(x_0,3R/2), |\nabla^j\eta| \leq CR^{-j}, j = 1,2$.

Note that

$$\begin{aligned} u(x)\eta(x) &= \int_{\mathbf{R}^n}\Gamma_0(x,y;\lambda)(L_0+\lambda)(u\eta)(y)\mathrm{d}y \\ &= \int_{\mathbf{R}^n}\Gamma_0(x,y;\lambda)(-Vu\eta)(y)\mathrm{d}y - \int_{\mathbf{R}^n}\Gamma_0(x,y;\lambda)A\nabla\eta\cdot\nabla u\mathrm{d}y + \\ &\quad \int_{\mathbf{R}^n}A\nabla_y\Gamma_0(x,y;\lambda)\nabla\eta u\mathrm{d}y. \end{aligned}$$

For $x \in B(x_0,R)$,

$$\begin{aligned} |\nabla u(z)| &= \int|\nabla_x\Gamma_0(x,y;\lambda)||V(y)u(y)\eta(y)|\mathrm{d}y + \\ &\quad \int|\nabla_x\Gamma_0(x,y;\lambda)||A\nabla\eta\cdot\nabla u|\mathrm{d}y + \int|A\nabla_x\nabla_y\Gamma_0(x,y;\lambda)\nabla\eta u|\mathrm{d}y. \end{aligned}$$

By Proposition 5, we use Caccioppoli's inequality (see [5], P.21) to estimate the second term, and notice that $|x-y| > R/2$ on the region $\{3R/2 < |x-x_0| < 2R\}$. It follows that

$$|\nabla u(x)| \leq \int_{B(x_0,3/2R)} \frac{V(y)|u(y)|}{|x-y|^{n-1}}\mathrm{d}y + \frac{C}{R^{n+1}}\int_{B(x_0,2R)}|u(y)|\mathrm{d}y. \quad \square$$

Now we are in the position to give our main theorems.

Theorem 7 *Suppose (A1)–(A3) for $A(x)$ and $V(x) \in B_q$ with $n/2 \leq q < n$. Then for $1 < p \leq p_0$,*

$$\|\nabla(L_0+V)^{-1/2}f\|_{L^p(\omega^{1-p})} \leq C_p\|f\|_{L^p(\omega^{1-p})} \text{ if } \omega \in A_{\frac{p'}{p_0'}},$$

where $\frac{1}{p_0}=\frac{1}{q}-\frac{1}{n}$. *And* $\nabla(L_0+V)^{-1/2}$ *is bounded from* H^1 *to* L^1.

Proof By functional calculus, we may write

$$(L_0+V)^{-1/2}=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(L_0+V+\lambda)^{-1}\mathrm{d}\lambda.$$

Thus

$$Tf(x)=\nabla(L_0+V)^{-1/2}f(x)=\int_{\mathbf{R}^n}K(x,y)f(y)\mathrm{d}y, \tag{3}$$

where

$$K(x,y)=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\partial_x\Gamma(x,y;\lambda)\mathrm{d}\lambda. \tag{4}$$

To prove Theorem 7, by duality it's equivalent to prove that $T^*f(x)=\int_{\mathbf{R}^n}K(y,x)f(y)\mathrm{d}y$ is bounded on $L^p(\omega)$ with $p_0'\le p<\infty$.

Let

$$T_0f(x)=\nabla L_0^{-1/2}f(x)=\int_{\mathbf{R}^n}K_0(x,y)f(y)\mathrm{d}y$$

where $K_0(x,y)=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\nabla_x\Gamma_0(x,y;\lambda)\mathrm{d}\lambda$.

We write

$$\begin{aligned}T^*f(x)=&T_0^*f(x)+\int_{|y-x|>r}K(y,x)f(y)\mathrm{d}y+\\&\int_{|y-x|\le r}(K(y,x)-K_0(y,x))f(y)dy-\int_{|y-x|>r}K_0(y,x)f(y)\mathrm{d}y\\:=&I_1+I_2+I_3+I_4,\end{aligned} \tag{5}$$

where $r=\frac{1}{m(x,V)}$.

First we estimate I_1, fix $x_0,y_0\in\mathbf{R}^n$. Let $u(y)=\Gamma(y,x_0;\lambda)$ and $R=|x_0-y_0|/4$. It follows from Lemma 2 that

$$|\nabla u(y_0)|\le C\int_{B(y_0,2R)}\frac{V(y)|u(y)|}{|y_0-y|^{n-1}}\mathrm{d}y+\frac{C}{R^{n+1}}\int_{B(y_0,2R)}|u(y)|\mathrm{d}y.$$

Hence, by Proposition 5,

$$\begin{aligned}&|\nabla_y\Gamma(y_0,x_0,\lambda)|\\&\le\frac{C_k}{(1+\lambda^{1/2}R)^k(1+Rm(x_0,V))^k}\{\int_{B(y_0,2R)}\frac{V(y)}{|y-y_0|^{n-1}}\frac{1}{|y-x_0|^{n-2}}\mathrm{d}y+\\&\quad\frac{1}{R^{n+1}}\int_{B(y_0,2R)}\frac{1}{|y-x_0|^{n-2}}\mathrm{d}y\}\\&\le\frac{C_k}{(1+\lambda^{1/2}R)^k(1+Rm(x_0,V))^k}(\frac{1}{R^{n-2}}\int_{B(y_0,2R)}\frac{V(y)}{|y-y_0|^{n-1}}\mathrm{d}y+\frac{1}{R^{n-1}}).\end{aligned}$$

Thus, by (4)

$$\begin{aligned}|K(y_0,x_0)|&\le C\int_0^\infty\lambda^{-1/2}|\nabla_y\Gamma(y_0,x_0;\lambda)|\mathrm{d}\lambda\\&\le C\frac{C_k}{(1+m(x_0,V)R)^k}\{\frac{1}{R^{n-2}}\int_{B(y_0,2R)}\frac{V(y)}{|y-y_0|^{n-1}}\mathrm{d}y+\frac{1}{R^n}\}.\end{aligned}$$

By the assumption $V(x) \in B_q$ for some $q, n/2 < q < n$, we know that there exists $q_1, q < q_1 < n$, such that $V(x) \in B_{q_1}$. We can get by the same strategy as [10] that

$$|I_2| \leq C\{M(|f|^{p_1'}(x)\}^{\frac{1}{p_1'}},$$

where $\frac{1}{p_1'} = 1 - \frac{1}{p_1} = 1 - \frac{1}{q_1} + \frac{1}{n}$, M is Hardy -Littlewood maximal function. Similarly, we have

$$|I_3| \leq C\{M(|f|^{p_1'}(x)\}^{\frac{1}{p_1'}}.$$

In [1], using homogenization theory, Alexopoulos obtained that the Riesz transforms associated to L_0 are C-Z operators when $A(x)$ has real-valued Hölder continuous coefficients that are periodic with common period. Hence by the standard C-Z theory,

$$\|T^* f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)} \quad \text{if} \quad \omega \in A_{\frac{p}{p_0'}}$$

for $p_0' \leq p < \infty$. And $T^* f$ is bounded from L^∞ to BMO. Therefore the proof is completed by duality. □

Theorem 8 *Suppose (A1) and (A2) for* $A(x), V(x) \in B_\infty$. *Then there exists constant* $C > 0$ *such that*

$$|V(x)^{1/2}(L_0 + V)^{-1/2} f(x)| \leq CMf(x), \quad f \in C_0^\infty(\mathbf{R}^n).$$

Proof We write

$$\begin{aligned} Sf(x) &= V^{1/2}(L_0 + V)^{-1/2} f(x) \\ &= \frac{V^{1/2}}{\pi} \int_0^\infty \lambda^{-1/2}(L_0 + V + \lambda)^{-1} f(x) \mathrm{d}\lambda \\ &= \frac{1}{\pi} \int_{\mathbf{R}^n} V(x)^{1/2} \int_0^\infty \lambda^{-1/2} \Gamma(x, y; \lambda) \mathrm{d}\lambda f(y) \mathrm{d}y. \end{aligned}$$

Let $r = \frac{1}{m(x,V)}$. By Proposition 5,

$$\begin{aligned} |Sf(x)| &\leq C_k \int_{\mathbf{R}^n} \frac{V(x)^{1/2}|f(y)|}{(1 + m(x,V)|x-y|)^k |x-y|^{n-1}} \mathrm{d}y \\ &\leq C_k \int_{\mathbf{R}^n} \frac{m(x,V)|f(y)|}{(1 + m(x,V)|x-y|)^k |x-y|^{n-1}} \mathrm{d}y \\ &= C_k \sum_{j \in \mathbf{Z}} \int_{2^{j-1}r < |x-y| \leq 2^j r} \frac{|f(y)|}{r(1 + m(x,V)|x-y|)^k |x-y|^{n-1}} \mathrm{d}y \\ &\leq C_k \sum_{j \in \mathbf{Z}} \frac{(2^j)^n}{(1 + 2^{j-1})^k (2^{j-1})^{n-1}} \frac{1}{(2^j r)^n} \int_{|x-y| \leq 2^j r} |f(y)| \mathrm{d}y \\ &\leq C_k \sum_{j \in \mathbf{Z}} \frac{2^j}{(1 + 2^j)^k} Mf(x) \leq CMf(x), \end{aligned}$$

where we choose $k > 2$. □

Corollary 9 *Let $A(x), V(x)$ be as in Theorem 8. Then $V^{1/2}(L_0 + V)^{-1/2}$ are bounded on $L^p(\omega)$ if $\omega \in A_p$.*

The corollary extends the result in [9,Theorem 5.10] to uniformly elliptic operators with general potentials $V \in B_\infty$. For $V(x) \in B_q$, with $q \geq n/2$, we have the following result.

Theorem 10 *Suppose (A1) and (A2) for $A(x)$, $V(x) \in B_q, n/2 \leq q < \infty$. Then, for $1 < p < 2q$,*

$$\|V(x)^{1/2}(L_0 + V)^{-1/2} f(x)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)},$$

if $\omega^{1-p'} \in A_{\frac{p'}{(2q)'}}$.

From the proof of Theorem 8, we have

$$|V(x)(L_0 + V)^{-1/2} f(x)| \leq C_k \int_{\mathrm{R}^n} \frac{V(x)^{1/2}|f(y)|}{(1 + m(x, V)|x - y|)^k |x - y|^{n-1}} dy.$$

Then the proof of this theorem is the same as that of Theorem 5.10 in [9].

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由 Schrödinger 算子定义的 Riesz 变换的 L^p 估计

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摘 要: 本文主要讨论了当非负位势 $V(x)$ 属于某逆 Hölder 类时，由一致椭圆算子 $L = -\mathrm{div}(A(x)\nabla) + V(x)$ 所定义的 Riesz 变换在 L^p 空间的有界性.

关键词: Riesz 变换； Schrödinger 算子.