Journal of Mathematical Research & Exposition
Vol.24, No.2, 239-248, May, 2004

Generalization of Ore's Theorem *

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Abstract: Let G be a graph. The partially square graph G^* of G is a graph obtained from G by adding edges uv satisfying the conditions $uv \notin E(G)$, and there is some $w \in N(u) \cap N(v)$, such that $N(w) \subseteq N(u) \cup N(v) \cup \{u, v\}$. In this paper, we will use the technique of the vertex insertion on l-connected ($l = k$ or $k+1, k \geq 2$) graphs to provide a unified proof for G to be hamiltonian, 1-hamiltonian or hamiltonian-connected. The sufficient conditions are expressed by the inequality concerning $\sum_{i=1}^{k} |N(Y_i)|$ and $n(Y)$ in G for each independent set $Y = \{y_1, y_2, \cdots, y_k\}$ in G^*, where $Y_i = \{y_i, y_{i-1}, \cdots, y_{i-(b-1)}\} \subseteq Y$ for $i \in \{1, 2, \cdots, k\}$ (the subscriptions of y_j's will be taken modulo k), b $(0 < b < k)$ is an integer, and $n(Y) = |\{v \in V(G) \ : \ \text{dist}(v, Y) \leq 2\}|$.

Key words: hamiltonicity; neighborhood union; vertex insertion; partially square graph.

Classification: AMS(2000) 05C35, 05C38/CLC number: O157.5

Document code: A　　**Article ID:** 1000-341X(2004)02-0239-10

1. Introduction

In this paper, the terminology and notation not defined will follow [2], and we consider simple finite graphs only. G will always stand for a graph.

Let $t > 1$ be an integer. Denote

$$I_t(G) = \{Y : Y \text{ is an independent set of } G, |Y| = t\}.$$

Let G be connected, $Y \subseteq V(G)$, and $v \in V(G)$. Denote $\text{dist}(v, Y) = \min_{y \in Y}\{\text{dist}(v, y)\}$ (where $\text{dist}(v, y)$ stands for the distance between v and y),

$$N_i(Y) = \{v \in V(G) : \ \text{dist}(v, Y) = i\} \ \ (i = 0, 1, 2, \cdots), \text{ and}$$

$$n(Y) = |N_0(Y) \cup N_1(Y) \cup N_2(Y)| = |\{v \in V(G) : \text{dist}(v, Y) \leq 2\}|.$$

*Received date: 2001-09-17
Foundation item: Supported by National Natural Sciences Foundation of China (19971043)
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For each $i \in \{0, 1, 2, \cdots, |Y|\}$, denote

$$S_i(Y) = \{v \in V(G) : |N(v) \cap Y| = i\}.$$

Clearly, $N(Y) = N_1(Y) = \bigcup_{i=1}^{|Y|} S_i(Y)$, and $n(Y) = |V(G) \setminus \bigcup_{l>2}(N_l(Y))| \leq |V(G)|$. For $v \in V(G)$, denote $N[v] = N(v) \cup \{v\}$. Let $\{u, v\} \subseteq V(G)$. Set

$$J(u, v) = \{w \in N(u) \cap N(v) : N(w) \subseteq N[u] \cup N[v]\}.$$

The partially square graph G^*[1] of G is a graph satisfying $V(G^*) = V(G)$ and $E(G^*) = E(G) \cup \{uv : uv \notin E(G), \text{ and } J(u, v) \neq \emptyset\}$.

In this paper, we will prove the following new results (Theorems 1–3) by using the vertex inserting lemmas introduced in [3]. In Theorems 1–3, we always assume that $Y = \{y_1, y_2, \cdots, y_k\} \in I_k(G)$, b is a given integer, $0 < b < k$,

$$Y_i = \{y_i, y_{i-1}, \cdots, y_{i-(b-1)}\} \subseteq Y$$

for $i \in \{1, 2, \cdots, k\}$ (where the subscriptions of y_j's will be taken modulo k).

Theorem 1 *Let G be a $k-$connected graph with $k \geq 2$; b an integer $(0 < b < k)$. If*

$$\sum_{i=1}^{k} |N(Y_i)| > \frac{b-1+k}{2}(n(Y) - 1)$$

in G for each $Y \in I_k(G^*)$, then G is hamiltonian. □

Theorem 2 *Let G be a $(k+1)-$connected graph with $k \geq 2$; b an integer $(0 < b < k)$. If*

$$\sum_{i=1}^{k} |N(Y_i)| > \frac{b-1+k}{2} n(Y)$$

in G for each $Y \in I_k(G^*)$, then G is 1-hamiltonian (if $G - w$ is hamiltonian for any $w \in V(G)$). □

Theorem 3 *Let G be a $(k+1)$-connected graph with $k \geq 3$; b an integer $(0 < b < k)$. If*

$$\sum_{i=1}^{k} |N(Y_i)| > \frac{b-1+k}{2} n(Y)$$

in G for each $Y \in I_k(G^*)$, then G is hamiltonian-connected. □

Clearly, Theorem 1 improves and generalizes Ore's Theorem[4].

We will use, in additional, the following notations.

Sometimes, by a slight abuse of notation, we shall use the same letter for a subgraph (of G) and its vertex set, provided no ambiguity arises.

Let U and R be subgraphs of G (or subsets of $V(G)$), denote $N_R(U) = N(U) \cap R$.

Each cycle or path of G discussed in this paper will be assigned an orientation. Let B be a cycle or path of G, $\{x, y\} \subseteq V(B)$, denote by $B[x, y]$ the oriented (x, y)-path of B

(where the orientation was taken from B), $B(x,y] = B[x,y] - \{x\}$, $B[x,y) = B[x,y] - \{y\}$, and $B(x,y) = B[x,y] - \{x,y\}$.

2. The vertex inserting lemmas and other lemmas

In this section, we always assume that G is a connected non-hamiltonian graph and C is a maximal cycle of G (i.e., there is no cycle C' in G such that $V(C) \subset V(C')$), and H is a component of $G - V(C)$. Assume also $\{v_1, v_2, \cdots, v_m\} \subseteq N_C(H)$ and $v_1, v_2, \cdots, v_m$ occur on C in the order of their indices. The subscriptions of v_i's will be taken modulo m. If $x \in V(C)$, denote by x^+ and x^- the successor and the predecessor of x along the orientation of C, respectively.

For each $i \in \{1, 2, \cdots, m\}$, a vertex $u \in C(v_i, v_{i+1})$ is called *insertible* [3], if there is some vertex $w \in C[v_{i+1}, v_i)$ such that $\{w, w^+\} \subseteq N(u)$. Otherwise u is called *non-insertible*.

Lemma 1[3] *Let $u \in C(v_i, v_{i+1})$ for some $i \in \{1, 2, \cdots, m\}$. If all the vertices in $C(v_i, u)$ are insertible, then $u \notin N_C(H)$. Therefore there exists a vertex in $C(v_i, v_{i+1})$, which is non-insertible.* □

By Lemma 1, for each $i \in \{1, 2, \cdots, m\}$, let x_i be the first non-insertible vertex in $C(v_i, v_{i+1})$.

Let $X_m = \{x_1, x_2, \cdots, x_m\}$, $X_M = X_m \cup \{x_0\}$ (where x_0 is an arbitrary vertex of H). Set

$$X = \{x_{p_1}, x_{p_2}, \cdots, x_{p_k}\} \subseteq X_m$$

where $1 \leq p_1 < p_2 < \cdots < p_k \leq m$. For convenience, we always assume that $x_{p_t} = x'_t$ and $v_{p_t} = v'_t$ for $t \in \{1, 2, \cdots, k\}$. Thus $X = \{x'_1, x'_2, \cdots, x'_k\}$. Denote $J_X = \bigcup_{t=1}^{k} C[x'_t, v'_{t+1}]$, $K_X = V(G) \setminus J_X$.

Lemma 2[3] *$X_M \in I_{m+1}(G)$, $X \in I_k(G)$, $K_X \subseteq S_0(X) \cup S_1(X)$, and $K_X \cap N_0(X) = \emptyset$.* □

Lemma 3[1] *$X_M \in I_{m+1}(G^*)$, and therefore $X \in I_k(G^*)$.* □

A segment $C[z_1, z_2)(\subseteq C[x'_t, v'_{t+1}], t \in \{1, 2, \cdots, k\})$ is called a CX-segment, if

(i) $C(z_1, z_2) \cap S_0(X) = \emptyset$, and

(ii) $z_1 \in N_2(X) \cup X, z_2 \in S_0(X) \cup \{v'^{+}_{t+1}\}$.

A CX-segment $C[z_1, z_2)$ is said to be *simple* if $C(z_1, z_2) \subseteq S_1(X)$.

Lemma 4[3] *Let $C[z_1, z_2)(\subseteq C[x'_t, v'_{t+1}], t \in \{1, 2, \cdots, k\})$ be a CX-segment. Set $L_i = N(x'_i) \cap C(z_1, z_2)$ $(i \in \{1, 2, \cdots, k\})$. Then*

$$L_t, L_{t-1}, \cdots, L_1, L_k, L_{k-1}, \cdots, L_{t+1}$$

(some of them may be empty) form consecutive subpaths of $C(z_1, z_2)$ which can have only their endvertices in common, and $|L_i| \leq 1$ for $i \in \{1, 2, \cdots, k\} \setminus \{t\}$. □

We always assume that b is an integer$(0 < b < k)$;

$$X_i = \{x'_i, x'_{i-1}, \cdots, x'_{i-(b-1)}\}(\subseteq X)$$

(for $i \in \{1, 2, \cdots, k\}$, and the subscriptions of x'_j's will be taken modulo k).

Let $U \subseteq V(G)$. We always set

$$\sigma_b(U, X) = \sum_{i=1}^{k} |N(X_i) \cap U|;$$

$$\sigma_b(X) = \sigma_b(V(G), X) = \sum_{i=1}^{k} |N(X_i)|.$$

Lemma 5 *(1) If $w \in S_1(X) \cap N(x'_q)$, then $\sigma_b(\{w\}, X) = b = b - 1 + |\{q\}|$.*

(2) Let

$$w \in S_{i_0}(X) \cap C[x'_t, v'_{t+1}] \cap N(x'_{q_1}) \cap N(x'_{q_2}) \cap \cdots \cap N(x'_{q_{i_0}})(i_0 \geq 2),$$

where

$$(t \geq) q_1 > q_2 > \cdots > q_{i'_0} (\geq 1)(k \geq) q_{i'_0+1} > \cdots > q_{i_0} (\geq t+1).$$

Then

$$\sigma_b(\{w\}, X) \leq \min\{k, b - 1 + |\{q_1, q_1 - 1, q_1 - 2, \cdots, q_{i_0}\}|\},$$

where the $q_1, q_1 - 1, q_1 - 2, \cdots, q_{i_0}$ are taken modulo k.

Proof (1) If $w \in S_1(X) \cap N(x'_q)$, then $w \notin N(x'_i)$ for any $i \in \{1, 2, \cdots, k\} \setminus \{q\}$. Thus $w \in \bigcap_{i=q}^{q+(b-1)} N(X_i)$ (where the subscriptions of X_i's will be taken modulo k), and

$$\sigma_b(\{w\}, X) = \sum_{i=q}^{q+(b-1)} |N(X_i) \cap \{w\}| = \sum_{i=q}^{q+(b-1)} 1 = b.$$

Therefore (1) holds.

(2) Clearly, $\sigma_b(\{w\}, X) = \sum_{i=1}^{k} |N(X_i) \cap \{w\}| \leq k$.

It is easy to see that

$$w \in \bigcup_{j=1}^{i_0} \bigcap_{i=q_j}^{\min\{q_{j-1}-1, q_j+(b-1)\}} N(X_i),$$

where $q_0 = q_1 + b$, the subscriptions of X_i's will be taken modulo k. Thus it is not difficult to see that (2) holds. □

Lemma 6 *(1) $\sigma_b(K_X, X) = b(|K_X| - |\bigcup_{l \geq 2}(N_l(X) \cap K_X)|)$.*

(2) Let $C[z_1, z_2)(\subseteq C[x'_t, v'_{t+1}], t \in \{1, 2, \cdots, k\})$ be a CX-segment. Then

$$\sigma_b(C[z_1, z_2), X) \leq \frac{b - 1 + k}{2} |C[z_1, z_2)|.$$

And

$$\sigma_b(C[z_1, z_2), X) = b(|C[z_1, z_2)| - 1)$$

when $C[z_1, z_2)$ is simple.

Proof Note that $0 < b < k$, and that b, k are integers, we have $b \le \frac{b-1+k}{2}$.

(1) By Lemma 2, $K_X \subseteq S_0(X) \cup S_1(X)$, and $K_X \cap N_0(X) = \emptyset$. Thus by Lemma 5(1), we have

$$\sigma_b(K_X, X) = b(|K_X| - |\bigcup_{l \ge 2}(N_l(X) \cap K_X)|)$$

so the (1) holds.

(2) By Lemma 4, we may assume $C(z_1, {z'_1}^+) = N(x'_t) \cap C[z_1, z_2)$, and $C(z_1, z'_1) \subseteq S_1(X)$. Let $W' = C(z_1, z'_1)$, $|W'| = h'$;

$$W = C[z'_1, z_2) = \{w_1, w_2, \cdots, w_h\},$$

$(w_1, w_2, \cdots, w_h$ occur on C in the order of their indices), and $w_j \in S_{i_j}(X)$; $h_1 = |W \cap S_1(X)|$, and $h_2 = h - h_1$ $(|C[z_1, z_2)| = h' + h + 1 = h' + h_1 + h_2 + 1)$. Thus there exist $x'_{q_1^{(j)}}, x'_{q_2^{(j)}}, \cdots, x'_{q_{i_j}^{(j)}} (i_j \ge 1)$, such that $w_j \in \bigcap_{l=1}^{i_j} N(x'_{q_l^{(j)}})$.

Notice that $C[z_1, z_2)$ is a simple CX-segment if and only if $h_2 = 0$. Thus if $C[z_1, z_2)$ is a simple CX-segment, then by Lemma 5(1),

$$\sigma_b(C[z_1, z_2), X) = b(h' + h_1) = b|C(z_1, z_2)| = b(|C[z_1, z_2)| - 1).$$

Therefore we may assume that $C[z_1, z_2)$ is not a simple CX-segment, so $h \neq 0$.

By Lemma 4,

$$\begin{aligned}(t \ge) q_1^{(1)} > q_2^{(1)} > \cdots > q_{i_1}^{(1)} > q_1^{(2)} > q_2^{(2)} > \cdots > q_{i_2}^{(2)} > \cdots \\ > q_1^{(j')} > q_2^{(j')} > \cdots > q_{i'_{j'}}^{(j')} (\ge 1, k \ge) q_{i'_{j'}+1}^{(j')} > \cdots > q_{i_{j'}}^{(j')} > \\ \cdots > q_1^{(h)} > q_2^{(h)} > \cdots > q_{i_h}^{(h)} (\ge t+1).\end{aligned}$$

Thus by Lemma 5, it is not difficult to see that

$$\begin{aligned}\sigma_b(C[z_1, z_2), X) &= \sigma_b(W \cup W', X) \\ &\le (h(b-1) + k) + bh' = bh' + (h+1)(b - 1 + \frac{k - (b-1)}{h+1}) \\ &\le bh' + (h+1)(b - 1 + \frac{k-(b-1)}{2}) \\ &\le \frac{b-1+k}{2}(h' + h + 1) = \frac{b-1+k}{2}|C[z_1, z_2)|,\end{aligned}$$

thus (2) holds. □

Lemma 7 *If there are λ simple CX-segments on C, then*

$$\sigma_b(X) \le \frac{b-1+k}{2}(n(X) - |N_2(X) \cap K_X| - \lambda).$$

Proof Consider that $X = \{x'_1, x'_2, \cdots, x'_k\}$ and $J_X = \bigcup_{t=1}^{k} C[x'_t, v'_{t+1}]$. For $t \in \{1, 2, \cdots, k\}$, partition

$$C[x'_t, v'_{t+1}] \setminus \bigcup_{l>2} (N_l(X) \cap C[x'_t, v'_{t+1}])$$

into s_t CX-segments

$$C[z_{11}^{(t)}, z_{12}^{(t)}), \ C[z_{21}^{(t)}, z_{22}^{(t)}), \ \cdots, \ C[z_{s_t 1}^{(t)}, z_{s_t 2}^{(t)}),$$

in which there are λ_t simple CX-segments

$$C[z_{11}, z_{12}), \ C[z_{21}, z_{22}), \ \cdots, \ C[z_{\lambda_t 1}, z_{\lambda_t 2})$$

and the other non-simple CX-segments

$$C[z'_{11}, z'_{12}), \ C[z'_{21}, z'_{22}), \ \cdots, \ C[z'_{s_t-\lambda_t 1}, z'_{s_t-\lambda_t 2}).$$

Set

$$\rho_t = \sum_{i=1}^{\lambda_t} |C[z_{i1}, z_{i2})|, \quad \rho = \sum_{t=1}^{k} \rho_t; \quad \lambda = \sum_{t=1}^{k} \lambda_t.$$

So

$$\sum_{j=1}^{s_t-\lambda_t} |C[z'_{j1}, z'_{j2})| = |C[x'_t, v'_{t+1}]| - \rho_t - |\bigcup_{l>2} (N_l(X) \cap C[x'_t, v'_{t+1}])|.$$

By Lemma 6(2), we have

$$\begin{aligned}
\sigma_b(C[x'_t, v'_{t+1}], X) &= \sum_{j=1}^{s_t} \sigma_b(C[z_{j1}^{(t)}, z_{j2}^{(t)}), X) \\
&= \sum_{j=1}^{s_t-\lambda_t} \sigma_b(C[z'_{j1}, z'_{j2}), X) + \sum_{j=1}^{\lambda_t} \sigma_b(C[z_{j1}, z_{j2}), X) \\
&\leq \sum_{j=1}^{s_t-\lambda_t} \frac{b-1+k}{2} |C[z'_{j1}, z'_{j2})| + \sum_{j=1}^{\lambda_t} b(|C[z_{j1}, z_{j2})| - 1) \\
&= \frac{b-1+k}{2} (|C[x'_t, v'_{t+1}]| - \rho_t - |\bigcup_{l>2} (N_l(X) \cap C[x'_t, v'_{t+1}])|) + b(\rho_t - \lambda_t).
\end{aligned}$$

Notice that $J_X = \bigcup_{t=1}^{k} C[x'_t, v'_{t+1}]$, $\rho = \sum_{t=1}^{k} \rho_t$; and $\lambda = \sum_{t=1}^{k} \lambda_t$. Thus

$$\begin{aligned}
\sigma_b(J_X, X) &= \sum_{t=1}^{k} \sigma_b(C[x'_t, v'_{t+1}], X) \\
&\leq \sum_{t=1}^{k} (\frac{b-1+k}{2} (|C[x'_t, v'_{t+1}]| - \rho_t - |\bigcup_{l>2} (N_l(X) \cap C[x'_t, v'_{t+1}])|) + b(\rho_t - \lambda_t)) \\
&= \frac{b-1+k}{2} (|J_X| - \rho - |\bigcup_{l>2} (N_l(X) \cap J_X)|) + b(\rho - \lambda).
\end{aligned}$$

Notice that $V(G) = J_X \cup K_X$, $n(X) = |V(G) \setminus \bigcup_{l>2}(N_l(X))|$, and $b \leq \frac{b-1+k}{2}$. So by Lemma 6(1), we have

$$\begin{aligned}\sigma_b(X) =&\sigma_b(J_X, X) + \sigma_b(K_X, X)\\ \leq&\frac{b-1+k}{2}(|J_X| - \rho - |\bigcup_{l>2}(N_l(X) \cap J_X)|) + b(\rho - \lambda)+\\ &b(|K_X| - |N_2(X) \cap K_X| - |\bigcup_{l>2}(N_l(X) \cap K_X)|)\\ =&\frac{b-1+k}{2}(|J_X| - \rho - |\bigcup_{l>2}(N_l(X) \cap J_X)|)+\\ &b(\rho - \lambda + |K_X| - |N_2(X) \cap K_X| - |\bigcup_{l>2}(N_l(X) \cap K_X)|)\\ \leq&\frac{b-1+k}{2}(|V(G) \setminus \bigcup_{l>2} N_l(X)|) - |N_2(X) \cap K_X| - \lambda)\\ =&\frac{b-1+k}{2}(n(X) - |N_2(X) \cap K_X| - \lambda).\end{aligned}$$

Lemma 8 *Let $i_0 \in \{1, 2, \cdots, k\}$. If $v'_{i_0} \in S_0(X)$, then $\lambda_{i_0-1} \geq 1$; if $v'_{i_0} \notin S_0(X)$, then there exists $x \in N_2(X) \cap K_X \cap N(v'_{i_0})$. So $|N_2(X) \cap K_X| + \lambda_{i_0-1} \geq 1$ (where λ_{i_0-1} be the number of simple CX-segments in $C[x'_{i_0-1}, v'_{i_0}]$).*

Proof If $v'_{i_0} \in S_0(X)$, then there exists an simple CX-segment $C[z, z^+)$ in $C[x'_{i_0-1}, v'_{i_o}]$ (where $z^- \in C[x'_{i_0-1}, v'_{i_0}]$ is the last vertex not in $S_0(X)$), so $\lambda_{i_0-1} \geq 1$; if $v'_{i_0} \notin S_0(X)$, then there exists $x \in N(v'_{i_0}) \cap V(H)$, such that $x \in N_2(X) \cap K_X$. So $|N_2(X) \cap K_X| + \lambda_{i_0-1} \geq 1$. □

Now we consider a graph G' other than G. In order to distinguish the notations such as $N(U)$, $S_i(X)$, $N_j(X)$, K_X, $n(X)$, $\sigma_b(X)$ introduced for G, we will simply add a prime to the notations with respect to G'. For example, $N'(U)$, $S'_i(X)$, etc.

By the proof of Theorems 9, 10 in [5], we have the following two Lemmas.

Lemma 9 *Assume that G be a $(k+1)$-connected graph with $k \geq 2$, and there exists some $w \in V(G)$ such that $G' = G - w$ is non-hamiltonian. Choose a cycle C of G' such that*

(i) $|N'_C(w)|$ is maximum; and

(ii) subject to (i), C is maximal.

Let H be a component of $G' - V(C)$, and $N'_C(H) = \{v_1, v_2, \cdots, v_m\}$ ($m \geq 3$)with the convention that $v_1, v_2, \cdots, v_m$ occur on C in the order of their indices. Set x_i as the first non-insertible vertex in $C(v_i, v_{i+1})$ for each $i \in \{1, 2, \cdots, m\}$. Then $X_M = \{x_0, x_1, \cdots, x_m\} \in I_{m+1}((G')^*)$ (where x_0 is an arbitrary vertex of H), and there exists $X \subseteq X_M \setminus \{x_0\}$ such that $X \in I_k(G^*)$. □

Lemma 10 *Assume that G is a connected non-hamiltonian-connected graph, there is some $\{u_1, u_2\} \subseteq V(G)$ such that G contains no (u_1, u_2)-hamiltonian-path, and there exists a (u_1, u_2)-path P such that*

(i) $V(P) \supseteq N(u_2)$;

(ii) subject to (i), $|N_P(u_1)|$ is maximum;

(iii) subject to (i), (ii), P is maximal.

Let H be a component of $G - V(P)$. Denote by G' the resulting graph obtained from G by adding a new vertex w and two new edges u_1w, u_2w. If $|N_P(H)| \geq k + 1 (k \geq 3)$, then

(1) In G', $C = P[u_1, u_2]wu_1$ is a maximal (choose the orientation of C agree with that of P), but not hamiltonian cycle of G'; H is a component of $G' - V(C)$.

(2) Let $\{v_1, v_2, \cdots, v_m\} = N'_C(H) = N_P(H)$. Then $v_m \neq u_2$, there exists the first non-insertible vertex x_i in $C(v_i, v_{i+1})$ for $i \in \{1, 2, \cdots, m\}$, where $m \geq k + 1 \geq 4$; $X_M = \{x_0, x_1, \cdots, x_m\} \in I_{m+1}((G')^*)$(where x_0 is an arbitrary vertex of H).

(3) There exists $X \subseteq X_M \setminus \{x_0\}$, such that $X \in I_k(G^*)$. □

3. Proofs of the theorems

Proof of Theorem 1 By contradiction. Suppose that G is non-hamiltonian. Let C be a longest cycle of G, and H a component of $G - V(C)$. Since G is a k-connected graph with $k \geq 2$, we have $|N_C(H)| \geq k$. Thus there exists $\{v_1, v_2, \cdots, v_k\} \subseteq N_C(H)$ (where $v_1, v_2, \cdots, v_k$ occur on C in the order of their indices, and $m = k$). By Lemma 1, for each $i \in \{1, 2, \cdots, k\}$, choose x_i the first non-insertible vertex in $C(v_i, v_{i+1})$. By Lemma 3, $X = \{x_1, x_2, \cdots, x_k\} \in I_k(G^*)$.

On the other hand, by Lemma 8, we have $|N_2(X) \cap K_X| + \lambda_1 \geq 1$ (where λ_1 be the number of simple CX-segments in $C[x_1, v_2]$, and $\lambda \geq \lambda_1$.). Thus by Lemma 7 , it is easy to see that

$$\sigma_b(X) = \sum_{i=1}^{k} |N(X_i)| \leq \frac{b-1+k}{2}(n(X) - |N_2(X) \cap K_X| - \lambda) \leq \frac{b-1+k}{2}(n(X) - 1),$$

a contradiction. □

Proof of Theorem 2 By contradiction. Suppose there exists some $w \in V(G)$ such that $G' = G - w$ is non-hamiltonian. Choose a cycle C of G' such that

(i) $|N'_C(w)|$ is maximum; and

(ii) subject to (i), C is maximal.

Let H be a component of $G' - V(C)$. Since G is $(k+1)$-connected with $k \geq 2$, $|N'_C(H)| \geq k$. Let $N'_C(H) = \{v_1, v_2, \cdots, v_m\}$ $(m \geq k)$ with the convention that $v_1, v_2, \cdots, v_m$ occur on C in the order of their indices. Set x_i as the first non-insertible vertex in $C(v_i, v_{i+1})$ for each $i \in \{1, 2, \cdots, m\}$. Let $X_m = \{x_1, , x_2 \cdots, x_m\}$. By Lemma 9, there is $X \subseteq X_m$, such that $X \in I_k(G^*)$.

On the other hand, note that

$$\sigma_b(\{w\}, X) = \sum_{i=1}^{k} |N(X_i) \cap \{w\}| \leq \xi k \leq \xi(b - 1 + k)$$

where $\xi = 0$ if $w \in S_0(X)$, otherwise $\xi = 1$. Clearly, $n'(X) \leq n(X) - \xi$. By Lemma 8, $|N'_2(X) \cap K'_X| + \lambda'_1 \geq 1$ (where λ'_1 are the number of simple CX-segments in $C[x_1, v_2]$ of

G'), and $\lambda' \geq \lambda'_1$. Thus by Lemma 7, we have

$$\begin{aligned}\sigma_b(X) &= \sum_{i=1}^{k}|N(X_i)| \leq \sum_{i=1}^{k}|N'(X_i)| + \xi(b-1+k) \\ &= \sigma'_b(X) + \xi(b-1+k) \\ &\leq \frac{b-1+k}{2}(n'(X) - |N'_2(X)\cap K'_X| - \lambda') + \xi(b-1+k) \\ &\leq \frac{b-1+k}{2}(n'(X) - 1 + 2\xi) \\ &\leq \frac{b-1+k}{2}(n(X) - \xi - 1 + 2\xi) \leq \frac{b-1+k}{2}n(X),\end{aligned}$$

a contradiction. □

Proof of Theorem 3 Suppose that graph G satisfies the conditions but is not hamiltonian-connected. Then there is some $\{u_1, u_2\} \subseteq V(G)$, and G contains no (u_1, u_2)-hamiltonian-path. By Theorem 2, there is a hamiltonian cycle C' in $G - u_2$. Choose an orientation of C'. Let $C'(u'_2, u_1) \cap N(u_2) = \emptyset$ and $u'_2 \in N(u_2)$, then the (u_1, u_2)-path $C'[u_1, u'_2]u_2$ contains the set $N(u_2)$. Thus one can choose a (u_1, u_2)-path P such that

(i) $V(P) \supseteq N(u_2)$;

(ii) subject to (i), $|N_P(u_1)|$ is maximum;

(iii) subject to (i), (ii), P is maximal.

Let H be a component of $G - V(P)$. Add a new vertex w and two new edges u_1w, u_2w to G and denote by G' the resulting graph. By Lemma 10(1), $C = P[u_1, u_2]wu_1$ is a maximal cycle in G' (choose the orientation of C agree with that of P), but not hamiltonian cycle of G'; H is a component of $G' - V(C)$. Let $\{v_1, v_2, \cdots, v_m\} = N'_C(H) = N_P(H)$. Since G is $(k+1)$-connected with $k \geq 3$, $m \geq k+1 \geq 4$. By Lemma 10(2), $v_m \neq u_2$, there exists the first non-insertible vertex x_i in $C(v_i, v_{i+1})$ for $i \in \{1, 2, \cdots, m\}$; $X_M = \{x_0, x_1, \cdots, x_m\} \in I_{m+1}((G')^*)$ (where x_0 is an arbitrary vertex of H). By Lemma 10(3), There exists $X \subseteq X_M \setminus \{x_0\}$ such that $X \in I_k(G^*)$.

On the other hand, by the construction of G', $n'(X) \leq n(X) + 1$. By Lemma 8, $|N'_2(X) \cap K'_X| + \lambda'_1 \geq 1$ (where λ'_1 be the number of simple CX-segments in $C[x_1, v_2]$ of G'), and $\lambda' \geq \lambda'_1$. Thus by Lemma 7, and the construction of G', it is easy to see that

$$\begin{aligned}\sigma_b(X) &= \sum_{i=1}^{k}|N(X_i)| \leq \sum_{i=1}^{k}|N'(X_i)| = \sigma'_b(X) \\ &\leq \frac{b-1+k}{2}(n'(X) - |N'_2(X)\cap K'_X| - \lambda') \\ &\leq \frac{b-1+k}{2}(n'(X) - 1) \leq \frac{b-1+k}{2}n(X),\end{aligned}$$

a contradiction. □

Acknowledgement The author appreciates the help of Prof. Wu Zheng-sheng.

References:

[1] AINOUCHE A, KOUIDER M. *Hamiltonism and partially square graphs* [J]. Graphs and Combinatorics, 1999, **15**(3): 257–265.
[2] BONDY J A, MURTY U S R. *Graph Theory with Applications* [M]. Macmillan, London and Elsevier, New York, 1976.
[3] LIU Y, TIAN F, WU Z. *Sequence concerning hamiltonicity of graphs* [J]. Nanjing Normal University (Natural Science), 1995, **18**(1): 19–28.
[4] ORE O. *Note on hamiltonian circuits* [J]. Amer. Math. Monthly, 1960, **67**: 55.
[5] WU Z, ZHANG X, ZHOU X. *Hamiltonicity, neighborhood intersections and partially squared graphs* [J]. Discrete Mathematics, 2002, **242**: 245–254.

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摘 要: 设 G 是一个图, G 的部分平方图 G^* 满足 $V(G^*)=V(G)$, $E(G^*)=E(G)\cup\{uv: uv\notin E(G)$, 且 $J(u,v)\neq\emptyset\}$, 这里 $J(u,v)=\{w\in N(u)\cap N(v): N(w)\subseteq N[u]\cup N[v]\}$. 本文利用插点方法, 给出了关于 k 或 $(k+1)$- 连通 $(k\geq 2)$ 图 G 是哈密尔顿的, 1- 哈密尔顿的或哈密尔顿连通的统一的证明。其充分条件是 G 中关于 $\sum_{i=1}^{k}|N(Y_i)|$ 与 $n(Y)$ 的不等式, 这里 $Y=\{y_1,y_2,\cdots,y_k\}$ 是图 G^* 的任一独立集, 对于 $i\in\{1,2,\cdots,k\}$, $Y_i=\{y_i,y_{i-1},\cdots,y_{i-(b-1)}\}\subseteq Y$ (y_j 的下标将取模 k) ; b 是一个整数, 且 $0<b<k$; $n(Y)=|\{v\in V(G): \mathrm{dist}(v,Y)\leq 2\}|$.

关键词: 哈密尔顿性; 邻域并; 插点; 部分平方图.