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Wide Diameters of Generalized Petersen Graphs *

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Abstract: Generalized Petersen graphs are an important class of commonly used interconnection networks and have been studied by various researchers. In this paper, we show that the diameter of generalized Petersen graph $P(m,2)$ is $O(\frac{m}{4})$ and the 3-wide diameter of $P(m,2)$ is $O(\frac{m}{3})$.

Key words: Petersen graph; generalized Petersen graph; diameter; wide diameter.

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1. Introduction

Let G be a graph with connectivity $k(G)$ and $w \le k(G)$. The w-wide diameter $d_w(G)$ of graph G is the minimum l such that for any two distinct vertices x and y there exist w vertex-disjoint paths of lengths at most l from x to y. This notion was introduced by Hsu[4] and was widely studied by various researchers.

Generalized Petersen graphs are an important class of commonly used interconnection networks and have been studied by various researchers(see [1,2,3,5]). The generalized Petersen graph, denoted by $P(m,a)$, is defined as follows: Its vertex set is $U \cup W$, where $U = \{u_0, u_1, \cdots, u_{m-1}\}, W = \{w_0, w_1, \cdots, w_{m-1}\}$, and its edges are given by $(u_i, w_i), (u_i, u_{i-1}), (u_i, u_{i+1})$, (w_i, w_{i+a}) and (w_i, w_{i-a}), for $0 \le i \le m-1$, with the addition of subscripts being performed under modulo m. We call the vertices in U as vertices in the outer circle and the vertices in W as the vertices in the inner circle. This generalization of Pertersen graph is due to Coxter[2].

Recently, Liaw and Chang[6,7,8] gave the wide diameter for many specific classes of networks. Krishnamoorthy and Krishnamurthy[5] gave the fault diameter for generalized Petersen graph. However, the analysis of its wide diameter is new. Observe that Petersen graph is $P(5,2)$. It is fairly straightforward to see that the connectivity of $P(m,1)$ is 3, the diameter of $P(m,1)$ is $[\frac{m}{2}]+1$, and the wide diameter of $P(m,1)$ is m (Consider the

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lengths of the vertex-disjoint paths between vertices u_0 and w_1). In this paper, we prove that the diameter of $P(m,2)$ is $O(\frac{m}{4})$ and the 3-wide diameter of $P(m,2)$ is $O(\frac{m}{3})$.

2. Main results

Let $P(m,a)$ be a generalized Petersen graph. It is clear that the connectivity of $P(m,a)$ is two when $m = 2a$ and $P(m,a) = P(m,m-a)$. So, in the following, we always suppose that $a < \frac{m}{2}$.

In [5], the authors gave the diameter of $P(m,2)$ when m is odd.

Lemma 1[5] *The diameter of $P(m,2) = O(\frac{m}{4})$, m is odd.*

When m is even, we have

Lemma 2 *The diameter of $P(m,2)$ is $O(\frac{m}{4})$, m is even.*

Proof If we consider only the outer cycle, the maximum distance between u_0 and $u_{\frac{m}{2}}$ is $\frac{m}{2}$. However, we can reach $u_{\frac{m}{2}}$ by first going to w_0 and going to in steps of 2 to $w_{\frac{m}{2}}$ and then going to $u_{\frac{m}{2}}$. Clearly this distance is $O(\frac{m}{4})$. (This is the case when $\frac{m}{2}$ even.) If $\frac{m}{2}$ is odd, go from u_0 to u_1, then to w_1 and through a W-path to $w_{\frac{m}{2}}$ and then to $u_{\frac{m}{2}}$.

This path strategy is fairly general to go from any vertex to any other vertex. The idea is to employ a W-path as much as possible, because it jumps in steps of 2. In order to show that this is, in fact, a lower bound, all we need to observe is that in a distance s less than $O(\frac{m}{4})$, the maximum vertex number we would have reached would be $2s$ and this implies that from u_0, we would not have reached $w_{\frac{m}{2}}$ and this contradicts the claim for diameter. □

It is clear that the connectivity of $P(m,2)$ is 3 if $m \neq 4$, as three vertex disjoint paths could be exhibited between any pair of vertices. In the following, we give the 3-wide diameter $d_3(P(m,2))$ when m is large enough.

Theorem 3 *The 3-wide diameter of $P(m,2) = O(\frac{m}{3})$, m is large enough.*

Proof Let x, y be any two distinct vertices of $P(m,2)$, there are two path strategies to choose the 3 vertex-disjoint paths from x to y. The difference of the two path strategies is that we employ the shortest U-path as the first path from x to y in the first strategy. We exhibit the path strategies when x and y are all in the outer circle. Suppose that $x = u_0$ and $y = u_i, i \le \frac{m}{2}$.

The first path strategy We choose the shortest path in the outer circle from x to y as the first path. If i is even, the 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - u_1 - u_2 \cdots u_{i-1} - u_i,$$

$$P_2 : u_0 - w_0 - w_2 \cdots w_i - u_i$$

and

$$P_3 : u_0 - u_{m-1} - w_{m-1} - w_1 - w_3 \cdots w_{i+1} - u_{i+1} - u_i$$

of lengths $|P_1| = i, |P_2| = 1 + \frac{i}{2} + 1 = \frac{i}{2} + 2$ and $|P_3| = 2 + 1 + \frac{i+1-1}{2} + 2 = \frac{i}{2} + 5$. It is clear that P_1 is the longest path when i is large enough. If i is odd, the 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - u_1 - u_2 \cdots u_{i-1} - u_i,$$

$$P_2 : u_0 - w_0 - w_2 \cdots w_{i+1} - u_{i+1} - u_i$$

and

$$P_3 : u_0 - u_{m-1} - w_{m-1} - w_1 - w_3 \cdots w_i - u_i$$

of lengths $|P_1| = i, |P_2| = 1 + \frac{i+1}{2} + 2 = \frac{i+1}{2} + 3$ and $|P_3| = 2 + 1 + \frac{i-1}{2} + 1 = \frac{i+1}{2} + 3$. And P_1 is the longest path when i is large enough.

The second path strategy Case 1. m is odd.

Case 1.1. i is even. There are 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - w_0 - w_2 \cdots w_i - u_i,$$

$$P_2 : u_0 - u_1 - w_1 - w_3 \cdots w_{i-1} - u_{i-1} - u_i$$

and

$$P_3 : u_0 - u_{m-1} - u_{m-2} - w_{m-2} - w_{m-4} \cdots w_{i+1} - u_{i+1} - u_i$$

of lengths $|P_1| = 1 + \frac{i}{2} + 1 = \frac{i}{2} + 2, |P_2| = 2 + \frac{i-2}{2} + 2 = \frac{i}{2} + 3$ and $|P_3| = 3 + \frac{m-2-i-1}{2} + 2 = \frac{m-i-1}{2} + 4$. It is clear that $|P_3|$ is the longest path since $i \leq \frac{m}{2}$.

Case 1.2. i is odd. We have 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - w_0 - w_2 \cdots w_{i-1} - u_{i-1} - u_i,$$

$$P_2 : u_0 - u_1 - w_1 - w_3 \cdots w_i - u_i,$$

and

$$P_3 : u_0 - u_{m-1} - w_{m-1} - w_{m-3} \cdots w_{i+1} - u_{i+1} - u_i$$

of lengths $|P_1| = 1 + \frac{i-1}{2} + 2 = \frac{i-1}{2} + 3, |P_2| = 2 + \frac{i-1}{2} + 1 = \frac{i-1}{2} + 3$ and $|P_3| = 2 + \frac{m-1-i-1}{2} + 2 = \frac{m-i}{2} + 3$. And P_3 is the longest path since $i \leq \frac{m}{2}$.

Case 2. m is even.

Case 2.1.i is even. There are 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - w_0 - w_2 \cdots w_i - u_i,$$

$$P_2 : u_0 - u_1 - w_1 - w_3 \cdots w_{i-1} - u_{i-1} - u_i$$

and

$$P_3 : u_0 - u_{m-1} - w_{m-1} - w_{m-3} \cdots w_{i+1} - u_{i+1} - u_i$$

of lengths $|P_1| = 1 + \frac{i}{2} + 1 = \frac{i}{2} + 2, |P_2| = 2 + \frac{i-2}{2} + 2 = \frac{i}{2} + 3$ and $|P_3| = 2 + \frac{m-1-i-1}{2} + 2 = \frac{m-i}{2} + 3$. It is clear that P_3 is the longest path since $i \leq \frac{m}{2}$.

Case 2.2. i is odd. There are 3 vertex-disjoint paths from x to y,

$$P_1 : u_0 - w_0 - w_2 \cdots w_{i-1} - u_{i-1} - u_i,$$

$$P_2 : u_0 - u_1 - w_1 - w_3 \cdots w_i - u_i,$$

and

$$P_3 : u_0 - u_{m-1} - u_{m-2} - w_{m-2} - w_{m-4} \cdots w_{i+1} - u_{i+1} - u_i$$

of lengths $|P_1| = 1 + \frac{i-1}{2} + 2 = \frac{i-1}{2} + 3, |P_2| = 2 + \frac{i-1}{2} + 1 = \frac{i-1}{2} + 3$ and $|P_3| = 3 + \frac{m-2-i-1}{2} + 2 = \frac{m-i-1}{2} + 4$. Clearly, P_3 is the longest path since $i \leq \frac{m}{2}$.

Compare the lengths of the longest paths of the two path strategies, for the case m is odd and i is even, let $\frac{m-i-1}{2} + 4 = i$, solve the equation, we get $i = \frac{m+7}{3}$. Then we choose the vertex-disjoint paths from x to y by the first path strategy if $1 \leq i \leq \frac{m+7}{3}$, by the second path strategy if $\frac{m+7}{3} < i \leq \frac{m}{2}$, that is we always have 3 vertex-disjoint paths of lengths at most $\frac{m+7}{3}$ from u_0 to u_i when m is odd and i is even. For the other cases, we can similarly get 3 vertex-disjoint paths of lengths at most $O(\frac{m}{3})$ from x to y.

For the cases x and y are all in the inner circle or one is in the outer circle and the other is in the inner circle, we can choose 3 vertex-disjoint paths from x to y of lengths at most $O(\frac{m}{3})$ similar to the case that x and y are all in the outer circle.

In the following, we show that $O(\frac{m}{3})$ is, in fact, the 3-wide diameter of $P(m, 2)$. By contradiction. Suppose that $d_3(P(m, 2)) = s < O(\frac{m}{3})$. Consider the two vertices u_0 and u_k, where $k = \frac{m+l}{3} (l = 0, 1, 2)$ such that k is an integer. Let P_1, P_2 and P_3 be 3 vertex-disjoint paths from u_0 to u_k with $|P_i| \leq s < O(\frac{m}{3})$. We first give two claims.

Claim 1 $P_i (i = 1, 2, 3)$ must not be the shortest U-path between u_0 and u_k.

This is clearly true since $k = \frac{m+l}{3} > s \geq |P_i|$.

Claim 2 $P_i (i = 1, 2, 3)$ must contain some vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$.

Because of Claim 1, P_i cannot be the shortest U-path from u_0 to u_k. In fact, if m and $k = \frac{m+l}{3} (l = 0, 1, 2)$ are even, the shortest path from u_0 to u_k which doesn't contain the vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$ and $\{u_1, u_2, \cdots, u_{k-1}\}$ is

$$P : u_0 - w_0 - w_{m-2} \cdots w_k - u_k$$

of length $|P| = 1 + \frac{m-k}{2} + 1 = \frac{m}{3} - \frac{l}{6} + 2 > \frac{m}{3} > s$. If $k = \frac{m+l}{3} (l = 0, 1, 2)$ is odd, the shortest path from u_0 to u_k which doesn't contain the vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$ and $\{u_1, u_2, \cdots, u_{k-1}\}$ is

$$P : u_0 - u_{m-1} - w_{m-1} - w_{m-3} \cdots w_k - u_k$$

of length $|P| = 1 + 1 + \frac{m-1-k}{2} + 1 = \frac{m}{3} + \frac{1}{2} - \frac{i}{6} + 2 > \frac{m}{3} > s$. For the case that m is odd, we can show that the lengths of the shortest path from u_0 to u_k which does not contain the vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$ and $\{u_1, u_2, \cdots, u_{k-1}\}$ is greater than s. This implies that the Claim 2 is true as well.

Let P_1 (P_2) be the path going from u_0, then to u_1 (w_0) and then going to u_k. Thus, P_1 and P_2 must contain some vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$ by Claims 1 and 2. Let P_3 be the path going from u_0 then to u_{m-1}, and then going to u_k. Since P_3 is vertex-disjoint to P_1 and P_2, P_3 dose not contain vertices of $\{w_1, w_2, \cdots, w_{k-1}\}$. This contradicts Claims 1 and 2.

We state a problem which might be worthy studying.

Problem Compute the diameter and wide diameter for $P(m, a)$ when $a > 2$.

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广义 Petersen 图的宽直径

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摘 要: 广义 Petersen 图是一类重要的并被广泛研究的互连网络。本文证明了广义 Petersen 图 $P(m,2)$ 的直径和 3 宽直径分别为 $O(\frac{m}{4})$ 和 $O(\frac{m}{3})$.

关键词: Petersen 图；广义 Petersen 图；直径；宽直径.