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Asymptotic Behavior of a Discrete-time Network Model of Two Neurons *

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Abstract: This paper is concerned with a delay difference systems arising from an artificial network model of two neurons with piecewise constant nonlinearity. The difference systems can be regarded as the discrete analog of the artificial neural network of two neurons. Some interesting results are obtained for the asymptotic behavior of solutions of the systems.

Key words: Neural networks; asymptotic behavior; difference system; piecewise constant nonlinearity.

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1. Introduction

Recently, there has been increasing interest in the study of the asymptotic behavior of solutions for delay difference equations and differential equations with piecewise constant argument. See, for example, [1–6] and the references cited therein.As mentioned in Cook and Wiener[1] and Shah and Wiener[2], the strong interest in such equations is motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.In this paper, we consider the following delay difference system:

$$\begin{cases} x_n = \lambda x_{n-1} + (1-\lambda)f(y_{n-k}), \\ y_n = \lambda y_{n-1} + (1-\lambda)f(x_{n-k}), \end{cases} \quad n = 1, 2, \cdots, \tag{1.1}$$

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where $\lambda \in (0,1)$, k is a positive integer, and $f : R \to R$ is a piecewise constant nonlinearity function given by

$$f(\xi) = \begin{cases} 1, & \xi \in (-\sigma, \sigma], \\ 0, & \xi \in (-\infty, -\sigma] \bigcup (\sigma, \infty) \end{cases} \tag{1.2}$$

for some constant $\sigma \in (0, \infty)$.

System (1.1) can be derived from the discrete analog of the following artificial neural network of two neurons with a piecewise constant argument

$$\begin{cases} \dot{x} = -\mu x + \beta f(y([t-l])), \\ \dot{y} = -\mu y + \beta f(x([t-l])), \end{cases} \tag{1.3}$$

where $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}$, $\mu > 0$ and $\beta > 0$ are given constants, l is an nonnegative integer, f is a signal function given by (1.2), $[t]$ denotes the greatest integer in t.

As we know, system (1.3) has also wide applications in certain biomedical models. For the backgrounds on system (1.3) and some other systems of differential equation involving piecewise constant argument, we refer to [3]. It is easy to convert (1.3) into a discrete system (1.1). In fact, we may rewrite (1.3) into the following form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(x(t)e^{\mu t}) = e^{\mu t}\beta f(y([t-l])), \\ \frac{\mathrm{d}}{\mathrm{d}t}(y(t)e^{\mu t}) = e^{\mu t}\beta f(x([t-l])). \end{cases} \tag{1.4}$$

Let n be a positive integer and $k = l+1$. Then we integrate (1.4) from $n-1$ to $t \in [n-1, n)$ to obtain

$$\begin{cases} x(t)e^{\mu t} - x(n-1)e^{\mu(n-1)} = \frac{\beta}{\mu}(e^{\mu t} - e^{\mu(n-1)})f(y(n-k)), \\ y(t)e^{\mu t} - y(n-1)e^{\mu(n-1)} = \frac{\beta}{\mu}(e^{\mu t} - e^{\mu(n-1)})f(x(n-k)). \end{cases} \tag{1.5}$$

Letting $t \to n$, we get from (1.5)

$$\begin{cases} x(n) = e^{-\mu}x(n-1) + \frac{\beta}{\mu}(1 - e^{-\mu})f(y(n-k)), \\ y(n) = e^{-\mu}y(n-1) + \frac{\beta}{\mu}(1 - e^{-\mu})f(x(n-k)). \end{cases} \tag{1.6}$$

Set $x_n^* = \frac{\mu}{\beta}x(n)$ and $y_n^* = \frac{\mu}{\beta}y(n)$ for any nonnegative integer n, $f^*(u) = f(\frac{\beta}{\mu}u), \sigma^* = \frac{\mu}{\beta}\sigma, \lambda = e^{-\mu}$, and then drop $*$ to obtain (1.1).

For the sake of simplicity, let N denote the set of all nonnegative integers. For any $a, b \in N$, define $N(a) = \{a, a+1, \cdots\}$ and $N(a,b) = \{a, a+1, \cdots, b\}$ whenever $a \le b$. In particular, $N = N(0)$. By a solution of (1.1), we mean a sequence $\{(x_n, y_n)\}$ of points in R^2 that is defined for all $n \in N(-k)$ and satisfies (1.1) for $n \in N$. Let X denote the set of mappings from $N(-k,-1)$ to R^2. Clearly, for any $\Phi = (\varphi, \psi) \in X$, system (1.1) has an unique solution $\{(x_n, y_n)\}$ satisfying the initial conditions

$$x_i = \varphi(i), \ y_i = \psi(i), \ \text{ for } \ i \in N(-k, -1). \tag{1.7}$$

Our goal is to determine the asymptotic behavior of $\{(x_n, y_n)\}$ as $n \to \infty$ for any $\Phi \in X_\sigma$. In particular, we concentrate on the case where $\varphi + \sigma, \varphi - \sigma$ and $\psi + \sigma, \psi - \sigma$ have no sign changes on $N(-k,-1)$. Namely, we consider those $\Phi \in \bigcup_{i,j=1}^{3} X_{ij} = X_\sigma$, where

$$X_{ij} = \{\Phi \in X; \Phi = (\varphi, \psi), \varphi \in R_i \ \text{ and } \ \psi \in R_j\}, \quad i, j = 1, 2, 3,$$

with

$$R_1 = \{\varphi; \varphi : N(-k,-1) \to R \text{ and } \varphi(i) \le -\sigma \text{ for } i \in N(-k,-1)\},$$

$$R_2 = \{\varphi; \varphi : N(-k,-1) \to R \text{ and } -\sigma < \varphi(i) \le \sigma \text{ for } i \in N(-k,-1)\}$$

and

$$R_3 = \{\varphi; \varphi : N(-k,-1) \to R \quad \text{and} \quad \varphi(i) > \sigma \quad \text{for} \quad i \in N(-k,-1)\}.$$

In this paper, we only consider the case where $\sigma \ge 1$. The case of $0 < \sigma < 1$ is considered in another paper. The main results of this paper are as follows

Theorem 1.1 *Let $\sigma > 1$. Then $(x_n, y_n) \to (1,1)$ as $n \to \infty$.*

Theorem 1.2 *Let $\sigma = 1$. Then the following conclusions hold.*

(I) If $\Phi \in X_{22}$, then $(x_n, y_n) \to (1,1)$ as $n \to \infty$;
(II) If $\Phi \in X_{23}$,then $(x_n, y_n) \to (0,1)$ as $n \to \infty$
(III) If $\Phi \in X_{32}$,then $(x_n, y_n) \to (1,0)$ as $n \to \infty$;
(IV) If $\Phi \in X_{33}$.Then
(i) $(x_n, y_n) \to (1,0)$ as $n \to \infty$, and $\varphi(-1) \ge \psi(-1)\lambda^{-k}$,
(ii) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $\psi(-1) < \varphi(-1) \le \psi(-1)\lambda^{1-k}$,
(iii) $(x_n, y_n) \to (0,1)$ as $n \to \infty$, and $\psi(-1) \ge \varphi(-1)\lambda^{-k}$,
(iv) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $\varphi(-1) < \psi(-1) \le \varphi(-1)\lambda^{1-k}$,
(v) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $\varphi(-1) = \psi(-1)$;
(V) If $\Phi \in X_{11} \bigcup X_{12} \bigcup X_{21}$, then $(x_n, y_n) \to (1,1)$ as $n \to \infty$;
(VI) If $\Phi \in X_{31}$, then
(i) $(x_n, y_n) \to (1,0)$ as $n \to \infty$, and $\varphi(-1) \ge -\psi(-1)\lambda^{-k}$,
(ii) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $-\psi(-1) < \varphi(-1) \le -\psi(-1)\lambda^{1-k}$,
(iii) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $-\psi(-1) \ge \varphi(-1)$;
(VII) If $\Phi \in X_{13}$, then
(i) $(x_n, y_n) \to (0,1)$ as $n \to \infty$, and $\psi(-1) \ge -\varphi(-1)\lambda^{-k}$,
(ii) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $-\varphi(-1) < \psi(-1) \le -\varphi(-1)\lambda^{1-k}$,
(iii) $(x_n, y_n) \to (1,1)$ as $n \to \infty$, and $-\varphi(-1) \ge \psi(-1)$.

2. Proofs of main results

Proof of Theorem 1.1 We distinguish several cases.

Case 1 Let $\Phi = (\varphi, \psi)^T \in X_{22}$.

In view of (1.1), we see that

$$\begin{cases} x_n = \lambda x_{n-1} + 1 - \lambda, \\ y_n = \lambda y_{n-1} + 1 - \lambda \end{cases} \tag{2.1}$$

for $n \in N(0, k-1)$. Therefore,

$$\begin{cases} x_n = (\varphi(-1) - 1)\lambda^{n+1} + 1, \\ y_n = (\psi(-1) - 1)\lambda^{n+1} + 1 \end{cases} \tag{2.2}$$

for $n \in N(0, k-1)$.

Note that

$$-\sigma < (-\sigma-1)\lambda^{n+1}+1 < x_n \le (\sigma-1)\lambda^{n+1}+1 < \sigma,$$
$$-\sigma < (-\sigma-1)\lambda^{n+1}+1 < y_n \le (\sigma-1)\lambda^{n+1}+1 < \sigma$$

for $n \in N(0, k-1)$, it follows that (2.1) is satisfied for $n \in N(k, 2k-1)$. Thus, $-\sigma < x_n \le \sigma, -\sigma < y_n \le \sigma$, for $n \in N(k, 2k-1)$. Repeating this procedure, we can obtain that (x_n, y_n) satisfies (2.2) for all $n \in N$, from which we know that $(x_n, y_n) \to (1,1)$ as $n \to \infty$.

Case 2 $\Phi \in X_\sigma - X_{22}$.

From (1.1), we see that

$$\begin{cases} 0 \le x_n - \lambda x_{n-1} \le 1-\lambda, \\ 0 \le y_n - \lambda y_{n-1} \le 1-\lambda \end{cases} \tag{2.3}$$

for $n \in N$. By induction, this implies

$$\begin{cases} \varphi(-1)\lambda^{n+1} \le x_n \le (\varphi(-1)-1)\lambda^{n+1}+1, \\ \psi(-1)\lambda^{n+1} \le y_n \le (\psi(-1)-1)\lambda^{n+1}+1, \end{cases} \quad n \in N. \tag{2.4}$$

Hence, we see that there exists a positive integer m_1 such that $-\sigma < x_n \le \sigma, -\sigma < y_n \le \sigma$ for $n \in N(m_1)$. Thus, by (1.1) and (1.2), we can obtain

$$\begin{cases} x_n = (x_{m_1+k-1}-1)\lambda^{n-m_1-k+1}+1, \\ y_n = (y_{m_1+k-1}-1)\lambda^{n-m_1-k+1}+1, \end{cases} \quad n \in N(m_1+k),$$

therefore $(x_n, y_n) \to (1,1)$ as $n \to \infty$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2 The conslusions (I),(II) and (III) can be proved by a similar argument as that in Case 1 of Theorem 1.1.

Now we consider (IV). Let $\Phi \in X_{33}$.

Case (i). $\varphi(-1) \ge \psi(-1)\lambda^{-k}$. From (1.1), we see that (x_n, y_n) satisfies

$$\begin{cases} x_n = \varphi(-1)\lambda^{n+1}, \\ y_n = \psi(-1)\lambda^{n+1} \end{cases} \tag{2.5}$$

for $n \in N(0, k-1)$. Let m_1 be the least nonnegative integer such that

$$y_{m_1-1} > 1, \ \ y_{m_1} \le 1.$$

Then (2.5) holds for $n \in N(0, m_1+k-1)$. By (2.5),

$$0 < y_{m_1+i} = \psi(-1)\lambda^{m_1+i+1} \le y_{m_1} \le 1,$$

and

$$\begin{aligned} x_{m_1+i} &= \varphi(-1)\lambda^{m_1+i+1} \ge \psi(-1)\lambda^{m_1+i+1-k} \\ &\ge \psi(-1)\lambda^{m_1} = y_{m_1-1} > 1, \ \ i \in N(0, k-1). \end{aligned}$$

Thus, $x_{m_1+i} > 1$, $-1 < y_{m_1+i} \le 1$ for $i \in N(0, k-1)$. By the conclusion (III), the case (i) can be proved.

Case (ii). $1 < \frac{\varphi(-1)}{\psi(-1)} \le \lambda^{1-k}$.

By the same argument as in that of (i), we have

$$0 < x_{m_1+k-1} = \varphi(-1)\lambda^{m_1+k} \le \psi(-1)\lambda^{m_1+k+1-k} = y_{m_1} \le 1,$$

and

$$0 < y_{m_1+k-1} = \psi(-1)\lambda^{m_1+k} < \varphi(-1)\lambda^{m_1+k} x_{m_1+k-1} \le 1,$$

which implies

$$-1 < x_n \le 1, \ \ -1 \le y_n \le 1, \ \text{for } n \in N(m_1+k-1).$$

Therefore (x_n, y_n) satisfies (2.1) for $n \in N(m_1+2k-1)$. By iteration, we have $(x_n, y_n) \to (1,1)$ as $n \to \infty$.

Case (iii) and Case (iv).

The proof of case (iii) follows from (i) and the symmetry of (1.1), and the proof of the case (iv) follows from (ii) and the symmetry of (1.1).

Case (v) $\varphi(-1) = \psi(-1)$.

Using (1.1), we can easily obtain that $x_n = y_n$ for all $n \in N$. Clearly, x_n satisfies equation

$$x_n = \lambda x_{n-1} + (1-\lambda) f(x_{n-k}) \tag{2.6}$$

with initial conditions $x_i \in R_3$ for $i \in N(-k, -1)$, and by (1.2), we have

$$x_n = \varphi(-1)\lambda^{n+1}, \ \ n \in N(0, k-1). \tag{2.7}$$

Assume that m_1 is the least nonnegative integer such that

$$x_{m_1-1} > 1, x_{m_1} \le 1.$$

Then (2.7) holds for $n \in N(0, m_1+k-1)$. By (2.7),

$$0 < x_{m_1+k-1} = \varphi(-1)\lambda^{m_1+k} \le \varphi(-1)\lambda^{m_1+1} = x_{m_1} \le 1,$$

which implies $-1 < x_{m_1+k-1}\lambda^{n-m_1-k+1} \le x_n \le (x_{m_1+k-1} - 1)\lambda^{n-m_1-k+1} + 1 \le 1$ for $n \in N(m_1+k-1)$. Therefore,

$$x_n = \lambda x_{n-1} + 1 - \lambda, \ n \in N(m_1+2k-1).$$

By iteration, we get $x_n \to 1$ as $n \to \infty$, and thus $(x_n, y_n) \to (1,1)$ as $n \to \infty$.

Next we prove conclusion (V). Let $\Phi \in X_{11} \bigcup X_{12} \bigcup X_{21}$.

We only prove the case of $\Phi \in X_{21}$. The cases with $\Phi \in X_{11}$ and $\Phi \in X_{12}$ are similar.

By (1.1) and (1.2), we see that

$$\begin{cases} x_n = \varphi(-1)\lambda^{n+1}, \\ y_n = [\psi(-1) - 1]\lambda^{n+1} + 1 \end{cases} \tag{2.8}$$

for $n \in N(0, k-1)$.

Assume that m_1 is the least nonnegative integer such that

$$y_{m_1-1} < -1,\ y_{m_1} \geq -1.$$

Then (2.8) holds for $n \in N(0, m_1+k-1)$. By (2.8),

$$\begin{cases} -1 < y_{m_1} \leq y_{m_1+i} = [\psi(-1)-1]\lambda^{m_1+i+1} + 1 < 1, \\ -1 < x_{m_1+i} = \varphi(-1)\lambda^{m_1+i+1} \leq 1, \end{cases} \quad i \in N(0, k-1),$$

by conclusion (I), conclusion(V) can be proved.

Conclusion (VI) and Conclusion (VII) can be proved in a fashion similar to Conclusion (IV). We omit the details.

This completes the proof of Theorem 1.2.

References:

[1] COOKE K L, WIENER J. *A Survey of Differential Equation with Piecewise Continuous Argument* [M]. Berlin, Springer-Verlag, 1991, 1–15.

[2] SHAH S M, WIENER J. *Advanced differential equations with piecewise constant argument deviations* [J]. Internat J. Math. Math. Sci., 1983, **6**: 671–703.

[3] AFTABIZADEH A R, WIENER J , XU J M . *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument* [J]. Proc. Amer. Math. Soc., 1987, **99**: 673–679.

[4] HUANG L H, WU J H. *Dynamics of inhibitory artificial neural networks with threshold nonlinearity* [J]. Fields Institute Communications, 2001, **29**: 235–243.

[5] HUANG L H, WU J H. *The role of threshold in preventing delay-induced oscillations of frustrated neural networks with McCulloch-Pitts nonlinearity* [J]. Int. J. Math. Game Theory Algebra, 2001, **11(6)**: 71–100.

[6] ZHU H Y, HUANG L H. *Dynamic analysis of a Discrete-Time network of two neurons with delayed feedback* [J]. Fields Institute Communications, 2004, **43**: 403–412.

一类二元离散神经网络模型的渐近性

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摘 要: 本文考虑的是一类具有分段常数非线性时滞差分系统, 该系统可作为二元人工神经网络模型的离散形式. 本文得到了系统解的渐近性的一些结果.

关键词: 神经网络; 渐近性; 差分系统; 分段常数非线性.