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Adaptive Finite Element Method Based on Optimal Error Estimtes for Linear Elliptic Problems on Concave Corner Domains (continuation) *

TANG Yan[1], *ZHENG Xuan*[2]

(1. School of Sciences, Tianjin University, Tianjin 300072, China;
2. Tianjin Institute of Urban Construction, Tianjin 300381, China)

Abstract: This paper is the third part in a series of papers on adaptive finite element methods based on optimal error estimates for linear elliptic problems on the concave corner domains. In this paper, a result is obtained. The algorithms for error control both in the energy norm and in the maximum norm presented in part 1 and part 2 of this series are based on this result.

Key words: adaptive finite element method; concave corner domain; elliptic problems.

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The purpose of this series of papers is to present an adaptive finite element method for the approximate solution of linear elliptic problems. The method is based on optimal a priori error estimate for the finite element method, and the regularity of the unknown exact solution is estimated in terms of difference quotients of computed approximate solutions.

This is the third part in a series of papers on adaptive finite element methods based on optimal error estimates for linear elliptic problems on the concave corner domains. In the preceding two papers[6,7], we presented an adaptive finite element method for the approximate solution of linear elliptic problems on concave corner domains. The algorithms that we presented in the Part 1 and Part 2 are based on the result of this paper.

The problem of constructing adaptive finite element methods is of great practical importance . For pioneering work we refer to [1], [2], [3].

Now we recall that our algorithm applied to this problem could generate in at most two steps of successive adaptive refinements a mash T and the corresponding approximate solution U so that $\|\nabla(u-U)\|_{L_2(\Omega)} \leq \delta$ for any given tolerance $\delta > 0$. And furthermore we had that $\|u-U\|_{L_\infty(\Omega)} \leq C(\delta \lg \frac{1}{\delta} + \delta^{1-(1-\beta/2)^{n+1}})$. The above methods that we got are based on a very important result that $|\nabla(u-U)| \leq Ch(x)|x|^{\beta-2}, |x| \geq C'\underline{h}$. In this paper, we will prove the above result.

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Biography: TANG Yan (1958-), male, Lecture.

1. A model problem

As a model problem, we will consider the Poisson equation

$$-\Delta u = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \Gamma, \tag{1}$$

where Ω has a concave corner with the interior angle π/β, $\frac{1}{2} \leq \beta < 1$.

We assume that f is a smooth function and that the vertex of the concave corner is located at the origin. We have[4]

$$|D^{\alpha}u(x)| \leq c|x|^{\beta-|\alpha|}, \quad x \in \Omega, |\alpha| \leq 3. \tag{2}$$

Furthermore, assume that for some positive constant C

$$|D^2 u(x)| \geq C|x|^{\beta-2}, \quad x \in \Omega, \tag{3}$$

so that the exact solution has a non-trivial singularity near the reentrant corner at the origin.

For the discretization of the problem and for triangles K of the partitions T of polygonal domain Ω, we will assume that

$$ch_K^2 \leq \int_K \mathrm{d}x, \quad \forall K \in T.$$

For triangulations T, we define the corresponding space of continuous piecewise linear functions vanishing on the boundary Γ of Ω

$$V_h = \{v \in C(\overline{\Omega}) : v|_K \in P_1(K), v|_\Gamma = 0, \forall K \in T\}$$

and the corresponding approximate solution $U \in V_h$ of the given model problem is defined by

$$a(U, v) = (f, v), \quad \forall v \in V_h, \tag{4}$$

where $a(w,v) = \int_\Omega \nabla w \cdot \nabla v \mathrm{d}x, (f,v) = \int_\Omega fv \mathrm{d}x$.

Under these assumptions, our algorithms will generate triangulations of Ω with the following characteristics: With $h > 0$, $0 \leq \alpha < 1$ and $d \geq h^{1/(1-\alpha)}$ for $x \in K \in T$, we have

$$h_K \sim h(x) = \begin{cases} h|x|^{\alpha}, & |x| \geq d, \\ hd^{\alpha}, & |x| < d, \end{cases} \tag{5}$$

where $h(x)$ is the local mesh size at x.

It is well known that[5]

$$\|v - v_i\|_{L_{P(K)}} + h_K \|\nabla(v - v_i)\|_{L_{P(K)}} \leq Ch_K^2 \|D^2 v\|_{L_{P(K)}}, \quad K \in T, \tag{6}$$

where $v_i \in V_h$ is the Lagrange interpolate of v. And we have proved the following[6],[7]

Theorem 1 *Let u be the solution of (1) satisfying (2) and (3), and $U \in V_h$ be determined* by (4) on a partition T of Ω as in (5) with $\alpha = 1 - \beta/2$. Then there is a constant C such that

$$|D^2u - D^{2,h}U| \le \frac{1}{2}D^2u, \quad |x| \ge C\underline{h}, \tag{7}$$

$$|D^{2,h}U| \le C\underline{h}^{\beta-2}, \quad |x| \le C\underline{h}, \tag{8}$$

where $\underline{h} = hd^{\alpha}$.

Theorem 2 *Let u be the solution of (1) satisfying (2). Under the assumptions of Theorem 1, there are constants C_1 and C such that*

$$|u(x) - U(x)| \le C_1 \begin{cases} \log \frac{1}{h(x)} h(x)^2 |x|^{\beta-2} + \underline{h}^{2\beta}|x|^{-\beta}, & |x| \ge C\underline{h}, \\ \underline{h}^{\beta}, & |x| \le C\underline{h}. \end{cases} \tag{9}$$

Our analysis for Theorems 1 and 2 is based on the following theorem

Theorem 3 *Let u be the solution of (1) satisfying (2), (3), and $U \in V_h$ be determined by (4) on a partition T of Ω as in (5) with $\alpha = 1 - \beta/2$. Then there are constants C and C' such that*

$$|\nabla(u(x) - U(x))| \le Ch(x)|x|^{\beta-2}, \quad |x| \ge C'\underline{h}. \tag{10}$$

In preparation for the proof of Theorem 3 we will give a useful estimate for the Green's function G associate with the boundary value problem (1).

Let $G(y,z)$ be the solution of

$$-\Delta G(y,z) = \delta(y-z) \ \text{ in } \ \Omega, \quad G(y,z) = 0 \ \text{ on } \ \Gamma,$$

where δ is the delta distribution.

Lemma *For domain Ω in (1), there is a constant C such that*

$$\left|\frac{\partial^2}{\partial y_k \partial y_j} G(y,z)\right| \le C \begin{cases} |y|^{\beta-2}|z|^{-\beta}, & |y| \le \frac{1}{2}|z|, \\ |y-z|^{-2}, & |y| > \frac{1}{2}|z|. \end{cases}$$

$$\left|\frac{\partial^3}{\partial y_k \partial y_j \partial y_i} G(y,z)\right| \le C \begin{cases} |y|^{\beta-2}|z|^{-\beta-1}, & |y| \le \frac{1}{2}|z|, \\ |y-z|^{-3}, & \frac{1}{2}|z| < |y| \le 2|z|, \\ |y|^{-\beta-2}|z|^{\beta-1}, & 2|z| < |y|. \end{cases}$$

It is easy to verify the above estimates. Now we consider the proof of Theorem 3.

2. Proof of Theorem 3

Proof Let $x \in K \in T$ and let δ_K an approximate delta function on K such that

$$\int_K \delta_K(z)\mathrm{d}z = 1,$$

and

$$\|\delta_K\|_{L_P(K)} \le C h_K^{-2+2/P}, \quad P = 1,2. \tag{11}$$

With $\partial_i = \frac{\partial}{\partial x_i}, i = 1,2$, we have[8]

$$\|\nabla(u-U)\|_{L_\infty(K)} \le Ch(x)|x|^{\beta-2} + \max_i |(u-U, \partial_i \delta_K)|, \tag{12}$$

In order to estimate the last term in (12) , let $\varphi = \varphi_i$ be solution of

$$-\Delta\varphi = \partial_i \delta_K, \quad \text{in} \quad \Omega, \quad \varphi = 0, \quad \text{on} \quad \Gamma, \quad i = 1,2. \tag{13}$$

From the representation

$$\varphi(y) = \int_\Omega G(y,z)\partial_i\delta_K(z)\mathrm{d}z = -\int_K \frac{\partial}{\partial z_i} G(y,z)\delta_K(z)\mathrm{d}z, \quad y \in \Omega, i = 1,2$$

where $G(y,z)$ is the associated Green's function as above. Notice

$$\frac{\partial^2}{\partial y_k \partial y_j}\varphi(y) = -\int_K \frac{\partial^3}{\partial y_k \partial y_j \partial z_i} G(y,z)\delta_K(z)\mathrm{d}z, \quad i,k,j = 1,2.$$

In view of (11) and lemma, we have

$$|D^2\varphi(y)| \le C \begin{cases} |y|^{\beta-2}|x|^{-\beta-1}, & |y| \le \frac{1}{2}|x|, \\ |y-x|^{-3}, & \frac{1}{2}|x| < |y| \le 2|x|, \\ |y|^{-\beta-2}|x|^{\beta-1}, & 2|x| < |y|. \end{cases} \tag{14}$$

From (14) and (11), we have

$$\|\nabla\varphi\|^2_{L_2(\Omega)} = (-\Delta\varphi, \varphi) = (\partial_i\delta_K, \varphi) = (\delta_K, -\partial_i\varphi) \le \|\delta_K\|_{L_2(K)}\|\nabla\varphi\|_{L_2(\Omega)}, \quad i = 1,2,$$

and hence

$$\|\nabla\varphi\|_{L_2(\Omega)} \le C h_K^{-1}. \tag{15}$$

Let $\varphi_h \in V_h$ be the Ritz projection of φ;

$$a(v, \varphi - \varphi_h) = 0, \quad v \in V_h$$

and put $e = \varphi - \varphi_h, \rho = u - u_i, u_i \in V_h$ is the Lagrange interpolate of u

$$(u-U, \partial_i\delta_K) = (u-U, -\Delta\varphi) = a(u-U, \varphi) = a(u-U, e) = a(\rho, e), \quad i = 1,2. \tag{16}$$

By the same arguments as in [5], we get

$$\begin{aligned} |a(\rho,e)| \le & C\Big(\sum_{j\in J_0} \|\nabla\rho\|_{L_\infty(C_j)} d_j \|\nabla e\|_{C_j} + \|\nabla\rho\|_{L_\infty(B_M)} d_M \|\nabla e\|_{B_M} + \\ & \sum_{j\in J_{12}} \|\nabla\rho\|_{L_\infty(E_j)} d_j \|\nabla e\|_{E_j} + \|\nabla\rho\|_{\Omega_I}\|\nabla e\|_{\Omega_I}\Big) \le C(\mathrm{I} + \mathrm{II}), \end{aligned} \tag{17}$$

$$
\begin{aligned}
\mathrm{I} &= h(x)|x|^{\beta-2}\sum_{j\in J_0} d_j\|\nabla e\|_{C_j} + \sum_{j\in J_{12}} h_j d_j^{\beta-1}\|\nabla e\|_{E_j} + \underline{h}^{\beta}\|\nabla e\|_{\Omega_I},\\
\mathrm{II} &= h(x)|x|^{\beta-2} d_M\|\nabla e\|_{B_M},\\
B_j &= \{y\in\Omega : |y-x| \le 2^{-j},\\
C_j &= \{y\in\Omega : 2^{-j} < |y-x| \le 2^{-j+1}\},\\
\Omega_j &= \{y\in\Omega\backslash B_{m+2} : |y| \le 2^{-j}\},\\
E_j &= \{y\in\Omega\backslash B_{m+2} : 2^{-j} < |y| \le 2^{-j+1}\}.
\end{aligned}
$$

We define $d_j = 2^{-j}, m$ is determined by $d_m < |x| \le 2d_m$ and

$$J_0 = \{j\in Z : m+3 \le j \le M\},\quad J_1 = \{j \in Z : j \le m\},$$

$$J_2 = \{j\in Z : m+1 \le j \le I\},\quad J_{12} = J_1 \cup J_2,$$

where $I, M \in Z$ and $C\underline{h} < d_I \le 2C\underline{h}$, $Ch(x) < d_M \le 2Ch(x)$.

We will show that for suitable choice of I and M

$$\|\nabla e\| \le C d_M, \tag{18}$$

$$\mathrm{I} \le \frac{1}{2}\mathrm{I} + C(h(x)|x|^{\beta-2} + h(x)|x|^{\beta-2} d_M\|\nabla e\|_{B_M} + \underline{h}^{2\beta}|x|^{-\beta-1}). \tag{19}$$

Combining (12), (16), (17), (18) and (19) yields estimate (10), since

$$\underline{h}^{2\beta} \le Ch(x)\underline{h}^{2\beta-1} \le Ch(x)|x|^{2\beta-1},\quad |x| \ge C\underline{h}.$$

By stability of the Ritz' projection, we have $\|\nabla e\| \le \|\nabla\varphi\|$, and (18) follows at once from (15).

For the proof of (19) we recall the local error estimate[6]

$$\|\nabla e\|_{D_j} \le C(\|\nabla(\varphi-\varphi_i)\|_{D_j'} + d_j^{-1}\|e\|_{D_j''}), \tag{20}$$

where $\varphi_i \in V_h$is the interpolate φ, D_j may be any one of the sets C_j, E_j, Ω_I and D_j' denotes the union of the sets D_j and its direct neighbors in the partition of Ω into the sets $C_j (j \ge m+3)$, $E_j (j\in J_{12})$, Ω_I and $D'' = D'\backslash D$.

According to (5), (6) (14), we have

$$\|\nabla(\varphi-\varphi_i)\|_{C_j'} \le Ch(x)d_j^{-2},\quad j\in J_0,$$

$$\|\nabla(\varphi-\varphi_i)\|_{E_j'} \le C\begin{cases} h_j d_j^{\beta-1}|x|^{-\beta-1}, & j\in J_2,\\ h_j d_j^{-\beta-1}|x|^{\beta-1}, & j \in J_1,\end{cases}$$

$$\|\nabla(\varphi-\varphi_i)\|_{\Omega_I'} \le C\underline{h}^{\beta}|x|^{-\beta-1}$$

By the fact $h(x) \le Cd_M$,$h(x) \le C|x|$, $\underline{h} \le Cd_I$, we find that

$$h(x)|x|^{\beta-2}\sum_{j\in J_0} d_j\|\nabla(\varphi-\varphi_i)\|_{C_j'} \le Ch(x)^2|x|^{\beta-2}\sum_{j\in J_0} d_j^{-1} \le Ch(x)^2|x|^{\beta-2}d_M^{-1} \le Ch(x)|x|^{\beta-2}$$

and

$$\sum_{j\in J_{12}} h_j d_j^{\beta-1}\|\nabla(\varphi-\varphi_i)\|_{E_j'} \le C(\sum_{j\in J_2} h_j^2 d_j^{2\beta-2}|x|^{-\beta-1}+\sum_{j\in J_1} h_j^2 d_j^{-2}|x|^{\beta-1})$$
$$\le C(\underline{h}^2 d_I^{2\beta-2}|x|^{-\beta-1}+h(x)^2|x|^{\beta-3}) \le C(\underline{h}^{2\beta}|x|^{-\beta-1}+h(x)|x|^{\beta-2})$$

using (20)

$$\mathrm{I} \le C(h(x)|x|^{\beta-2}+\underline{h}^{2\beta}|x|^{-\beta-1}+h(x)|x|^{\beta-2}\sum_{j\in J_0}\|e\|_{C_j}+\sum_{j\in J_{12}} h_j d_j^{\beta-2}\|e\|_{E_j}+\underline{h}^{\beta} d_I^{-1}\|e\|_{\Omega_I} \tag{21}$$

In order to estimate the term $\|e\|_{C_j}$, let $\psi=\psi_j$ be the solution of

$$-\Delta\psi=\varphi \ \text{ in } \ \Omega, \quad \psi=0 \ \text{ on } \ \Gamma, \tag{22}$$

where $\varphi=e/\|e\|_{C_j}$ in C_j, $\varphi=0$ in $\Omega\backslash C_j$. Again we will use a representation of solution in terms of the Green's function. We have [4]

$$\psi(y)=\gamma r^{\beta}\sin\beta\vartheta+\psi_i(y), \tag{23}$$

where $\gamma=\gamma_j=\int_{C_j}\xi(z)e(z)\mathrm{d}z/\|e\|_{C_j}$, $|\xi(z)|\le C|z|^{-\beta}$ with

$$|\gamma_j|\le Cd_j|x|^{-\beta} \tag{24}$$

and

$$\|\psi_i\|_{H^2(\Omega)}\le C. \tag{25}$$

By (5), (6), (23), (24), (25), we have

$$\|\nabla(\psi-\psi_i)\|_{C_k}\le Ch(x)\begin{cases} d_k d_j^{-1}, & k\in J_0, k\ge j,\\ d_k^{-1}d_j, & k\in J_0, k<j,\end{cases} \tag{26}$$

$$\|\nabla(\psi-\psi_i)\|_{B_M}\le Ch(x)d_M d_j^{-1}, \tag{27}$$

$$\|\nabla(\psi-\psi_i)\|_{E_k}\le Ch_k\begin{cases} d_k^{\beta-1}|x|^{-\beta}d_j, & k\in J_2,\\ d_k^{-1}d_j, & k\in J_1,\end{cases} \tag{28}$$

and finally, we get[6]

$$\|\nabla(\psi-\psi_i)\|_{\Omega_I}\le C\underline{h}^{\beta}|x|^{-\beta}d_j. \tag{29}$$

By the standard duality arguments and (26), (27), (28), (29), we have

$$\|e\|_{C_j}\le C(\sum_{k\in J_0,k\ge j} h(x)d_k d_j^{-1}\|\nabla e\|_{C_k}+\sum_{k\in J_0,k<j} h(x)d_k^{-1}d_j\|\nabla e\|_{C_k}+$$
$$h(x)d_M d_j^{-1}\|\nabla e\|_{B_M}+\sum_{k\in J_{12}} h_k d_k^{\beta-1}|x|^{-\beta}d_j\|\nabla e\|_{E_k}+\underline{h}^{\beta}|x|^{-\beta}d_j\|\nabla e\|_{\Omega_I},$$

and thus

$$Ch(x)|x|^{\beta-2}\sum_{j\in J_0}\|e\|_{C_j}\le \frac{1}{6}\mathrm{I}+Ch(x)|x|^{\beta-2}d_M\|\nabla e\|_{B_M}. \tag{30}$$

By the argument similar to used above, we find that

$$C\sum_{j\in J_{12}} h_j d_j^{\beta-2}\|e\|_{E_j} \le \frac{1}{6}\mathrm{I} + Ch(x)|x|^{\beta-2} d_M\|\nabla e\|_{B_M}, \tag{31}$$

$$C\underline{h}^{\beta} d_I^{-1}\|e\|_{\Omega_I} \le \frac{1}{6}\mathrm{I} + Ch(x)|x|^{\beta-2} d_M\|\nabla e\|_{B_M}. \tag{32}$$

The estimates (21), (30), (31), (32) complete the proof of the Theorem 3.

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自适应有限元方法在凹角域线性椭圆方程的应用 (续)

汤　雁[1]，郑　璇[2]

(1. 天津大学理学院，天津 300072; 2. 天津城建学院，　天津 300381)

摘 要: 作为序列文章自适应有限元方法在凹角域线性椭圆方程的应用的第三篇，在本文我们将给出并详细论证一个重要结论即

$$|\nabla(u(x)-U(x))| \le Ch(x)|x|^{\beta-2},\quad |x| \ge C'\underline{h}$$

且进一步分析说明在本序列文章的第一部分和地二部分得出方法都是以此为基础作出的.

关键词: 自适应有限元方法；凹角域；椭圆方程.