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Robustness of Minimum Norm Quadratic Unbiased Estimator of Variance for the General Linear Model *

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Abstract: In this paper, necessary and sufficient conditions for equalities between $a^2y'(I-P_X)y$ and $\tilde{\sigma}^2$ under the general linear model, where $\tilde{\sigma}^2=\frac{y'T^{\frac{1}{2}+}(I-P_{T^{1/2+}X})T^{\frac{1}{2}+}y}{\text{rank}T-\text{rank}X}$ and a^2 is a known positive number, are derived. Furthermore, when the Gauss-Markov estimators and the ordinary least squares estimators are identical, we obtain a simple equivalent condition.

Key words: general linear model; generalized inverse; orthogonal projector; minimum norm quadratic unbiased estimator.

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1. Introduction

This paper adopts the following notations:

Let $M_{m,n}$ denote the set of $m\times n$ real matrices, S_n be the subset of $M_{n,n}$ consisting of symmetric matrices, and $S_n^{\geq}$ be the subset of S_n consisting of nonnegative definite matrices. For $A,\ B\in M_{n,n}$, we will write $A\geq B$ whenever $A-B\in S_n^{\geq}$. Given $A\in M_{m,n}$, the symbols $A', R(A)$, rank$A$, A^- will stand for the transpose, the range, the rank, and the generalized inverse, respectively, of A. $R^{\perp}(A)$ will stand for the orthocomplement to $R(A)$. Let P_A denote the orthogonal projector matrix onto $R(A)$. The symbols $E(y)$ and $D(y)$ will stand for the mean and the variance respectively, of random vector y.

In recent years, the study of robustness of a statistical inference has become a very popular topic. Huber[4] gave a solid foundation of the concept of robustness from both the theoretical and the applied statistical viewpoints. Rousseeuw and Leroy[9] examined the properties of robustness in the linear regression case. Some robust properties of ordinary

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least squares estimators in linear models are also investigated. For example, we consider two linear models, denoted by

$$y = X\beta + e, \quad E(e) = 0, \quad D(e) = \sigma^2 I_n, \tag{1.1}$$

$$y = X\beta + e, \quad E(e) = 0, \quad D(e) = \sigma^2 \Sigma, \tag{1.2}$$

where y is an $n \times 1$ observable random vector, $X \in M_{n,p}$ is a non-null known matrix with $\mathrm{rank} X = r \le p$, $\beta \in M_{p,1}$ is an unknown parametric vector, σ^2 is a unknown positive scalar, $\Sigma \in S_n^{\geq}$ is a known matrix.

It is well known that GME, OLSE of the expectation vector are expressible as $X\tilde{\beta}$ and $X\hat{\beta}$, respectively, with

$$\tilde{\beta} = (X'T^+X)^+X'T^+y \tag{1.3}$$

and

$$\hat{\beta} = (X'X)^- X', \tag{1.4}$$

and necessary and sufficient conditions for equalities between GME and OLSE are given by many statisticians. For general background and bibliography see [1],[2],[7] and [12].

It is well known that $\hat{\sigma}^2$ and $\tilde{\sigma}^2$ are minimum norm quadratic unbiased estimator of σ^2 in model (1.1) and (1.2) (see also [8]), respectively, where $\hat{\sigma}^2 = \frac{1}{n-r}(y-X\hat{\beta})'(y-X\hat{\beta})$, $\tilde{\sigma}^2 = \frac{1}{\mathrm{rank}T - \mathrm{rank}X}(y-X\tilde{\beta})'T^+(y-X\tilde{\beta})$. Kruskal[5] showed that necessary and sufficient condition for $\hat{\sigma}^2$ to equal $\tilde{\sigma}^2$ when $X\hat{\beta} = X\tilde{\beta}$ holds is that Σ is the identity operator on $R^{\perp}(X)$. Zhang, Luo and Qiu[11] considered that necessary and sufficient condition for $\hat{\sigma}^2$ to equal $\tilde{\sigma}^2$ when $\Sigma \in M_{n,n}$ is a positive and symmetric matrix.

In this paper, we mainly establish that necessary and sufficient conditions for $a^2(y - X\hat{\beta})'(y - X\hat{\beta})$ to coincide with $\tilde{\sigma}^2$ of σ^2. As a result, works by Kruskal[5] and Zhang, Luo and Qiu[11] are extended.

2. Main results

To prove Theorem 2.1, we introduce the following three lemmas at first without proofs [8,10].

Lemma 2.1 *For $A,\ B \in M_{m,n}$, the following statements are equivalent:*
(1) $P_A = P_B$,
(2) $R(A) = R(B)$.

Lemma 2.2 *Let $A \in M_{n,m}, B \in M_{n,p}$ such that $R(B) \subseteq R(A)$, then $P_A - P_B$ is the orthogonal projector onto $R(A) \cap R(B^{\perp})$*

Lemma 2.3 *For $A,\ B \in S_n$, the following statements are equivalent:*
(1) $y'Ay = y'By$ for all $y \in R^n$,
(2) $A = B$.

Theorem 2.1 *In model (1.2), the following statements are equivalent:*
(1) $a^2(y - X\hat{\beta})'(y - X\hat{\beta}) = \tilde{\sigma}^2$;
(2) $a^2 f_A(I - P_X)\Sigma$ is idempotent;

(3) $a^2 f_A \Sigma = XB_1X' + XB_2Z' + ZB_2'X' + ZB_3Z'$. (2.1)

Here $Z = X^\perp$ is the matrix with maximum rank among all matrices which satisfy $X'X^\perp = 0$, $f_A = \text{rank}(T) - \text{rank}X$, B_1, B_2 and B_3 are arbitrary symmetric matrices such that $\Sigma \in S_n^{\geq}$, and ZB_3Z' is the orthogonal projector satisfying $(ZB_3Z')(ZB_2'X') = ZB_2X'$.

Proof Since $T \in S_n^{\geq}$, there exists unique $T^{\frac{1}{2}} \in S_n^{\geq}$ such that $T = T^{\frac{1}{2}} \cdot T^{\frac{1}{2}}$ (see e.g. Theorem 7.2.6 in [3]). In fact, by Lemma 2.3 (1) holds if and only if

$$a^2 f_A T^{\frac{1}{2}}(I - P_X)T^{\frac{1}{2}} = T^{\frac{1}{2}}(T^+ - T^+X(X'T^+X)^+X'T^+)T^{\frac{1}{2}} = P_T - P_{T^{\frac{1}{2}+}X}, \qquad (2.2)$$

and by Lemma 2.2, we obtain $P_T - P_{T^{1/2+}X}$ is the orthogonal projector onto

$$R(T) \cap R^\perp(T^{\frac{1}{2}+}X) = R(T^{\frac{1}{2}}X^\perp).$$

Note that

$$R(T^{\frac{1}{2}}(I - P_X)T^{\frac{1}{2}}) = R(T^{\frac{1}{2}}X^\perp).$$

So by Lemma 2.1, (1) holds if and only if

$$a^2 f_A T^{\frac{1}{2}}(I - P_X)T^{\frac{1}{2}} \text{ is idempotent},$$

which leads to

$$a^4 f_A{}^2 T^{\frac{1}{2}}(I - P_X)T(I - P_X)T^{\frac{1}{2}} = a^2 f_A T^{\frac{1}{2}}(I - P_X)T^{\frac{1}{2}},$$

or equivalently,

$$a^2 f_A(I - P_X)T \text{ is idempotent},$$

$$a^2 f_A(I - P_X)\Sigma \text{ is idempotent}.$$

This completes the proof of $(1) \Longleftrightarrow (2)$.

It follows from $a^2 f_A \Sigma \in S_n^{\geq}$ that there exists a matrix $Q \in M_{n,n}$ such that $a^2 f_A \Sigma = QQ'$, Q can be decomposed as

$$Q = XU_1 + ZU_2,$$

where U_1 and U_2 are some matrices. This gives

$$a^2 f_A \Sigma = XB_1X' + XB_2Z' + ZB_2'X' + ZB_3Z',$$

where $B_1 = U_1U_1'$, $B_2 = U_1U_2'$, $B_3 = U_2U_2'$. Since

$$(I - P_X)a^2 f_A \Sigma(I - P_X)a^2 f_A \Sigma(I - P_X) = (I - P_X)a^2 f_A \Sigma(I - P_X),$$

we have

$$ZB_3Z' = ZU_2U_2'Z'$$

is the orthogonal projector onto $R(ZU_2)$, and $(ZB_3Z')(ZB_2'X') = ZB_2X'$. This proves $(2) \Longrightarrow (3)$

Conversely, noting that

$$(ZB_3Z')(ZB_2'X') = ZB_2X',$$

the part of (3) $\Longrightarrow$ (2) is obvious. □

Theorem 2.2 *In model (1.2), If $X\hat{\beta} = X\tilde{\beta}$ holds, then the following statements are equivalent:*

(1) $a^2(y - X\hat{\beta})'(y - X\hat{\beta}) = \tilde{\sigma}^2$,

(2) $a^2 f_A(I - P_X)\Sigma$ *is the orthogonal projector onto* $R(\Sigma X^{\perp})$;

(3) $a^2 f_A\Sigma = XB_1X' + ZB_3Z'$,

where B_1 and B_3 are defined as before.

Proof Since $X\hat{\beta} = X\tilde{\beta}$, we have

$$(I - P_X)\Sigma(I - P_X) = (I - P_X)\Sigma = \Sigma(I - P_X), \tag{2.3}$$

which leads to

$$XB_2Z' = ZB_2'X' = 0.$$

Therefore (2.1) reduces to (3). The part of (1)$\Longleftrightarrow$ (3) is easily verified. By Theorem 2.1, we have

$$a^2 f_A(I - P_X)\Sigma \text{ is idempotent},$$

and

$$a^2 f_A(I - P_X)\Sigma = a^2 f_A(I - P_X)\Sigma(I - P_X)$$

is also symmetric. Hence, (2) holds, this completes the equivalence between (1) and (2). □

Corollary 2.1 *In model (1.2), if ΣM_X is the orthogonal projector onto $R(\Sigma X^{\perp})$, then* $X\tilde{\beta} = X\hat{\beta}$ and $(y - X\hat{\beta})'(y - X\hat{\beta}) = \tilde{\sigma}^2$.

Proof From Theorem 2.2, the proof is trivial.

Remark When $\Sigma \in M_{n,n}$ is a positive and symmetric matrix, we have U_1 and U_2 are the column full rank matrix. Hence conditions (2) and (3) of Theorem 2.1 are equivalent to "$a^2(n - r)\Sigma$ is a generalized inverse of $(I - P_X)$" and "$a^2(n - r)\Sigma = XB_2X' + XB_2Z' + ZB_2'X' + I - P_X$", respectively, which lead to Zhang, Luo and Qiu's Theorem 2.1 in [11]. Similarly, other results are extended.

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一般线性模型下方差最小范数二次无偏估计的稳健性

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摘 要: 本文给出了一般线性模型下，由最小二乘得到的方差估计与最小范数二次无偏估计相等的充分必要条件，并且当 Gauss-Markov 估计与最小二乘估计相等时，可以得到一个简单的等价条件.

关键词: 一般线性模型；广义逆；正交投影；最小范数二次无偏估计.