Journal of Mathematical Research & Exposition
Vol.24, No.3, 400–404, August, 2004

Regular Factor in Vertex Transitive Graphs *

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Abstract: Let G be a k-regular connected vertex transitive graph. If G is not maximal restricted edge connected, then G has a $(k-1)$-factor with components isomorphic to the same vertex transitive graph of order between k and $2k-3$. This observation strenghen to some extent the corresponding result obtained by Watkins, which said that k-regular vertex transitive graph G has a factor with components isomorphic to a vertex transitive graphs if G is not k connected.

Key words: vertex transitive graph; regular factor; restricted edge cut; fragment.

Classification: AMS(2000) 05C70/CLC O157.5

Document code: A **Article ID**: 1000-341X(2004)03-0400-05

1. Introduction

All graphs considered in this paper are undirected connected finite simple k-regular vertex transitive graphs with $k \geq 3$ if not specified. Restricted edge cut is such an edge cut that separates a connected graph into a disconnected one with no component having order less than two. The cardinality of a minimum restricted edge cut of graph G is called its restricted edge connectivity, and denoted by $\lambda'(G)$. Let $d(X)$ indicate the number of the edges with one end in X and the other in $G\backslash X$, define $\xi(G)=\min\{d(X): X$ is a connected subgraph of G with order two $\}$. Esfahanian proved in [1] that

$$\lambda'(G) \leq \xi(G) = 2k-2.$$

If the equality in the previous inequality holds, then G is called maximal restricted edge connected. With these notations and terminology we prove in this paper the following

Theorem 3.1 *Let G be a k-regular connected vertex transitive graph. If G is not maximal*

*Received date: 2001-09-25
Foundation item: Supported by NNSF of China (10271105) and Doctoral Foundation of Zhangzhou Normal College.
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restricted edge connected, then G has a $(k-1)$*-factor with all components isomorphic to the same vertex transitive graph of order* $\lambda'(G)$ *such that* $k \le \lambda'(G) \le 2k-3$.

This observation improves the counterparts obtained by Watkins[2], Jan van den Heuvel and Bill Jacson[3] respectively. The former claims that k-regular vertex transitive graph G has a factor F such that the components of F are isomorphic vertex transitive graphs if G is not k connected. The latter states that if vertex transitive graph G has an edge cut S with cardinality less than $2(k+1)^2/9$ such that no component of $G-S$ has order less than $(k+1)/3$, then G has a factor F such that the components of F are isomorphic vertex transitive graphs.

Before proceeding, let's introduce some more notations and terminology. Refer to the two components resulting from the removal of a minimum restricted edge cut S from graph G as restricted fragments, or simply fragments, corresponding to S, among which the smaller one (with less vertices) is called a normal fragment. Fragments with minimum cardinality are atoms. Clearly, normal fragments and atoms of graph G have order at most half that of G. It is worth noting that fragments defined here do not coincide the traditional ones, which are connected vertex induced subgraphs corresponding to some minimum restricted edge cut and appear in pairs. If we denote one fragment by X, then the other one appearing together with X is denoted by X^c. For simplicity, we signify a fragment with its vertex set. For two subsets A and B of $V(G)$ or two subgraphs of G, let $G \setminus A$ denote the graph obtained by removing the vertices of A from G, and $[A, B]$ represent the set of edges with one end in A and the other in B. We simplify $[A, G\setminus A]$ as $I(A)$, whose cardinality is denoted by $d(A)$. A regular factor of graph G is a regular spanning subgraph of G. A graph G is vertex transitive if for any two vertices $u, v \in V(G)$ there is an automorphism $\tau \in \mathrm{Aut}(G)$ with the property that $\tau(u) = v$, where $\mathrm{Aut}(G)$ is the automorphism group of graph G. Denote by $\lambda(G)$ the edge connectivity of graph G. Other symbols and terminologies not specified coincide with that in [5].

2. Properties of restricted fragments

In order to prove the main result in section three, we need to introduce some preliminaries in this section. Since the main result is obviously true for the vertex transitive graphs with valence one or two, we restrict ourselves to connected k-regular vertex transitive graphs with $k \ge 3$ in the rest of this paper.

Lemma 2.1[4] *Every connected k-regular vertex transitive graphs is k-edge connected.*

Lemma 2.2 *Let X and Y be two different fragments of a connected k-regular vertex transitive graph G. If* $|X \cap Y| \ge 2$ *and* $d(X \cap Y) \le \lambda'(G)$*, then* $X \cap Y$ *is a fragment.*

Proof Let X and Y be two fragments corresponding to minimum restricted edge cut S and T respectively. Write

$$A = X \cap Y,\ B = X \cap Y^c,\ C = X^c \cap Y,\ D = X^c \cap Y^c.$$

It suffices to show that both $X \cap Y$ and $X^c \cup Y^c$ are connected.

Claim 1 $X^c \cup Y^c$ is connected.

Since both X^c and Y^c are connected fragments, Claim 1 is true if D is not empty. We shall prove that $[B, C] \neq \emptyset$ when $D = \emptyset$, as a result Claim 1 is also true in the case when $D = \emptyset$. Suppose to the contrary that $[B, C] = \emptyset$ when $D = \emptyset$, then

$$d(A) = |[A, B]| + |[A, C]| = |S| + |T| = 2\lambda'(G) > \lambda'(G),$$

which contradicts Condition (2).

Claim 2 $X \cap Y$ is connected.

Let $A_i, i = 1, 2, \ldots, m$, be all the components of A. If Claim 2 fails, then $m \geq 2$. According to Lemma 2.1 and the fact that $\lambda'(G) \leq 2k - 2$, we have

$$2k - 2 \geq \lambda'(G) = d(A) = \sum_{i=1}^{m} d(A_i) = 2k.$$

This contradiction negates our hypothesis and confirms Claim 2.

Lemma 2.3 *Let X and Y be two different normal fragments of G. If $|X \cap Y| \geq 2$, then $X \cap Y$ is a fragment.*

Proof Let S and T be the two restricted edge cuts corresponding to X and Y respectively. Define A, B, C and D the same as in the proof of Lemma 2.2. According to Lemma 2.2, we need only to show that $\lambda'(G) \geq d(A)$.

(1) $|D| \geq |A| \geq 2$.

Since X and Y are two normal fragments of G, it follows that

$$|A| + |C| = |Y| \leq |G|/2 \leq |X^c| = |C| + |D|.$$

Statement (1) follows from this formula.

(2) $d(D) \geq \lambda'(G)$.

If D is connected, then $d(D)$ is a restricted edge cut by (1) since D^c is connected, Statement (2) thus follows in this case. If D is not connected, $d(D)$ contains at least two disjoint edge cut each of which has cardinality at least k. Therefore $d(D) \geq 2\lambda(G) = 2k > \lambda'(G)$.

(3) $\lambda'(G) \geq d(A)$.

Since

$$\begin{aligned} d(A) + d(D) =& |[A, C]| + |[A, B]| + |[D, C]| + |[D, B]| + 2|[A, D]| + 2|[B, C]| - 2|[B, C]| \\ \leq& |[A, C]| + |[A, B]| + |[D, C]| + |[D, B]| + 2|[A, D]| + 2|[B, C]| \\ =& |S| + |T| = 2\lambda'(G). \end{aligned}$$

Statement (3) follows from the combination of this formula and (2).

Corollary 2.4 *Let $X$and Y be two distinct atom of G. Then $|X \cap Y| \leq 1$.*

Proof Since atoms are normal fragments containing no fragments as their proper subgraphs (with less vertices), Corollary 2.4 follows from Lemma 2.3.

Lemma 2.5 *Let X and Y be two distinct atoms of G. If G is not maximal restricted edge connected, then $X \cap Y = \emptyset$.*

Proof Since G is not maximal restricted edge connected, it follows that $\lambda'(G) < 2k-2 = \xi(G)$ and $|X| = |Y| \geq 3$. Suppose by contradiction that $X \cap Y \neq \emptyset$. According to Corollary 2.4, $|X \cap Y| = |A| \leq 1$. Define A, B, C and D the same as in the proof of Lemma 2.2. Since

$$|A| + |B| = |X| = |Y| = |A| + |C| \geq 3,$$

it follows that

$$|B| = |C| \geq 2. \tag{1}$$

Since

$$d(B) + d(C) \leq 2\lambda'(G),$$

we have

$$d(B) \leq \lambda'(G) \quad \text{or} \quad d(C) \leq \lambda'(G). \tag{2}$$

Combining (1), (2) and Lemma 2.2, we conclude that B or C is a fragment. Hence X or Y contains a fragment of G different from itself, which is a contradiction.

Lemma 2.6 *Let X be an atom of G. If G is not maximal restricted edge connected, then X is vertex transitive.*

Proof Since G is a vertex transitive graph, for any two vertices $u, v \in X$ there is an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau(u) = v$. But $\tau(X)$ is also an atom of G, furthermore $\tau(X) \cap X \neq \emptyset$ since vertex $v \in X \cap Y$, hence $\tau(X) = X$ by Lemma 2.5. It follows that $\tau|_X$, the restriction of τ in X, is an automorphism of graph X. X is thus a vertex transitive graph.

3. Regular factor

Theorem 3.1 *Let G be a k-regular connected vertex transitive graph. If G is not maximal restricted edge connected, then G has a $(k-1)$-factor with all components isomorphic to the same vertex transitive graph of order $\lambda'(G)$ such that $k \leq \lambda'(G) \leq 2k-3$.*

Proof Let X be an arbitrary atom of G. According to Lemma 2.6, X is a vertex transitive graph with valence r. Since $I(X)$ is an restricted edge cut, it follows that $0 < r < k$. From the condition that G is not maximal restricted edge connected, we conclude that $|X| > 2$ and that

$$\begin{aligned} 2k - 2 > \lambda'(G) = d(X) &= k|X| - \sum_{u \in X} d_X(u) \\ &> k|X| - |X|(|X| - 1), \end{aligned}$$

where $d_X(u)$ denotes the valence of vertex u in X. From above formula, we have

$$(|X| - 2)(|X| - (k-1)) > 0.$$

Since $|X|-2>0$, it follows that $|X|\geq k$. Now we can see that

$$2k-2>\lambda'(G)=d(X)=k|X|-r|X|=(k-r)|X|\geq(k-r).$$

Since $0<k-r<k$, we have

$$r=k-1.$$

According to the last two formulas, we conclude that X is a $(k-1)$-regular vertex transitive graph with order $\lambda'(G)$ such that $k\leq\lambda'(G)\leq 2k-3$. Since $\tau(X)=X$ or $\tau(X)\cap X=\emptyset$ for any automorphism $\tau\in\mathrm{Aut}(G)$, it follows that $\bigcup\limits_{\tau\in Aut(G)}\tau(X)$ is a $(k-1)$-factor with all its components isomorphic to a vertex transitive graph with order $\lambda'(G)$ such that $k\leq\lambda'(G)\leq 2k-3$.

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点可迁图中的正则因子

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摘 要: 设图 G 是一个 k- 正则连通点可迁图. 如果 G 不是极大限制性边连通的，那么 G 含有一个 $(k-1)$- 因子，它的所有分支都同构于同一个阶介于 k 和 $2k-3$ 之间的点可迁图. 此结果在某种程度上加强了 Watkins 的相应命题：如果 k 正则点可迁图 G 不是 k 连通的，那么 G 有一个因子，它的每一个分支都同构于同一个点可迁图.

关键词: 点可迁图；正则因子；限制性边割；断片.