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Edge-Tenacity in Graphs *

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Abstract: The edge-tenacity of a graph $G(V,E)$ is defined as $\min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}: S\subseteq E(G)\}$,where $\tau(G-S)$ and $\omega(G-S)$, respectively, denote the order of the largest component and the number of the components of $G-S$. This is a better parameter to measure the stability of a network G, as it takes into account both the quantity and the order of components of the graph $G-S$. In a previous work, we established a necessary and sufficient condition for a graph to be edge-tenacious. These results are applied to prove that K-trees are strictly edge-tenacious. A number of results are given on the relation of edge-tenacity and other parameters, such as the higher-order edge toughness and the edge-toughness.

Key words: edge cut-sets; strictly edge-tenacious graph; K-trees; higher-order edge toughness; edge toughness.

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1. Introduction

One way of measuring the stability of a network(computer, communi-cation, or transportation) is through the cost with which one can disrupt the network. The edge-connectivity gives the minimum cost to disrupt the network, but it does not take into account what remains after disruption. One can say that the disruption is more successful if the disconnected network contains more components and much more successful, if in addition, the components are small. As nicely explained in [1], one can associate the cost with the number of edges destroyed to get small components and associate the reward with the number of components remaining after destruction. The edge-tenacity measure is a compromise between the cost and the reward by minimizing the cost: reward ratio. Thus, a network with a large edge tenacity performs better under external attack. In this

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sense, the following parameters are successively better for the measurement of stability. Before defining these parameters, we recall some standard notation and terminology.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, S a subset of $E(G)$. Denote by $\omega(G-S)$ the number of (connected)components of $G-S$, by $\tau(G-S)$the order(number of vertices)of a largest component of $G-S$. A subset $S \subseteq E(G)$ is an edge cut-set of G ,if $G-S$ is disconnected .

Edge-connectivity:

$$\lambda(G)=\min\{|S|: S\subseteq E(G) \text{ is an edge cut set of } G\};$$

Edge-toughness[3]:

$$\tau_1=\min\{\frac{|S|}{\omega(G-S)-1}; S\subseteq E(G) \text{ is an edge cut} - \text{set of } G\};$$

Cth-order edge toughness[4]:

$$\tau_c=\min\{\frac{|S|}{\omega(G-S)-c}: S\subseteq E(G)\&\omega(G-S)>c(1\le c\le |V(G)|-1)\};$$

Edge-tenacity[1]:

$$T'(G)=\min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}: S\subseteq E(G)\}.$$

The score of S is define as $sc(S)=[|S|+\tau(G-S)]/[\omega(G-S)]$. Formally, the edge-tenacity of a graph G is defined as $T'(G)=\min\{sc(S)\}$, where the minimum is taken over all edge sets S of G. Let $T^*(G)=\min\{sc(S)\}$, where the minimum is taken over all edge-sets of $S\neq E$of G. A subset $S\neq E$ of $E(G)$ is said to be a T^*-set of G if $T^*(G)=sc(S)$. Note that if G is disconnected, then the set S may be empty. Throughout this paper, we use ω and τ to represent $\omega(G-S)$ and $\tau(G-S)$, respectively, when $G$and $S$are clear from the context. We also use p and q to represent the number of vertices (order) and the number of edges (size), respectively, of a graph. The edge-connectivity of G will be denoted $\lambda=\lambda(G)$, definitions and notation not otherwise defined here can be found in [6].

A graph G is called edge-tenacious if $T'(G)=sc(E(G))$. A graph is called strictly edge-tenacious if $E(G)$ is the unique set whose score equals $T'(G)$.

2. Relationships with $\tau_c(G)(1\le c\le p-1)$

In this section, we shall recall some known results that will be used to investigate some connections between strictly tenacious graphs and $\tau_c(G)(1\le c\le p-1)$.

Lemma 1[2] *Let G be a graph, S a T^*-set of G. Assume that $G-S$ has at most ω' nontrivial components. If*

$$\frac{q_i}{p_i-1}<\frac{q+1}{p}+\frac{1}{\omega'}$$

for any nontrivial component H_i of $G-S(p_i=|V(H_i)|, q_i=|E(H_i)|, i=1,2,\ldots,k, k\le \omega')$, then G is strictly edge-tenacious.

Lemma 2[2] *If S is a T^*-set and C is a nontrivial component of $G-S$, then $\lambda(C) \geq T^*(G)$.*

Lemma 3[2] *Let G be connected and $S \subseteq E$. Then $sc(S) \geq 1$ with equality if and only if G is a tree and $S = E$.*

Lemma 4[4] *Let G be a graph, $s = \frac{q}{p-c}$, where c is an integer satisfying $1 \leq c \leq p-1$. Then $\tau_c(G) = s$ if and only if $|E(H)| \leq s(|V(H)|-1)$ for every subgraph of H of G.*

From Lemmas 1–4, we have

Theorem 1 *If G is a connected graph with $q < 4(p-1)(p \geq 2)$ and $\tau_1(G) = \frac{q}{p-1}$, then G is strictly edge-tenacious.*

Proof Let S be a T^*-set of G. Assume that $G-S$ has ω' nontrivial components $H_1, H_2, \cdots, H_t, t = \omega'(p_i = |V(H_i)|, q_i = |E(H_i)|, i = 1, 2, \cdots, t)$. It follows from Lemmas 2, 3 that $\lambda(H_i) \geq T^*(G) > 1$ and so $p_i \geq 3(i = 1, 2, \cdots, t)$. This implies that $p_1 + \cdots + p_t \geq 3\omega'$. Thus, $\omega' \leq \frac{p}{3}$.

From Lemma 4 and $q < 4(p-1)$, we know $\frac{q_i}{p_i-1} \leq \frac{q}{p-1} < \frac{q+1}{p} + \frac{3}{p}$, and thus

$$\frac{q_i}{p_i-1} < \frac{q+1}{p} + \frac{1}{\omega'}.$$

From Lemma 1, G is strictly edge-tenacious. □

From Lemma 4 and the proof of Theorem 1, we have the following results.

Theorem 2 *If G is a connected graph with $q < 2p-4(p > 3)$ and $\tau_2(G) = \frac{q}{p-2}$, then G is strictly edge-tenacious.*

Theorem 3 *If G is a connected graph with $q < \frac{4}{3}(p-3)(p > 9)$ and $\tau_3(G) = \frac{q}{p-3}$, then G is strictly edge-tenacious.*

3. K-trees

In this section, we study the edge-tenacity of k-trees. We prove that k-trees are strictly edge-tenacious.

We now define a k-tree. Let k be a positive integer. Then k-trees are graphs defined recursively as follows:

The smallest k-tree is the complete graph with k vertices ,and a k -tree with$n+1$ vertices, where $n \geq k$ is obtained by adding a new vertex adjacent to each of the k arbitrarily selected but mutually adjacent vertices of a k-tree with n vertices.

We need the following results to determine the edge-tenacity of k-trees.

Lemma 5[10] *Let G be a k-tree of order p and H be its sub-graph with p' vertices, where $p \geq p' \geq k+1$. Then $|E(H)| \leq p'k - k(k+1)/2$ and $|E(H)| = p'k - k(k+1)/2$ if and only if H is k-tree.*

We shall also need the following results:

Lemma 6[7] *If Gis k-tree with p vertices and q edges, then $\tau_1(G) = q/(p-1)$.*

Lemma 7[2] *Let G be a graph, and let S be a T^*-set of G. Assume that $G-S$ has at most one nontrivial component. If $\tau_1(G)=q/(p-1)$, then G is strictly edge-tenacious.*

Theorem 4 *If is G a k-tree, then G is strictly edge-tenacious.*

Proof Let S be a T^*-set of$G$. Assume that $G-S$ contains at least two nontrivial components H_1 and H_2 of order p_1 and p_2 , respectively, where, without loss of generality, $p_2 \geq p_1$.

In order to complete the proof we shall examine three different cases.

Case 1 $p_2 \geq p_1 \geq k+1$.

Then, by Lemma 5,

$$|E(H_1)| \leq p_1k - k(k+1)/2, |E(H_2)| \leq p_2k - k(k+1)/2.$$

Thus,

$$|S| \geq pk - k(k+1)/2 - [p_1k - k(k+1)/2 + p_2k - k(k+1)/2] = (p-p_1-p_2)k + k(k+1)/2.$$

Since $\omega \leq p-p_1-p_2+2, \tau \geq p_2$, we have

$$sc(S) \geq \frac{(p-p_1-p_2)k + k(k+1)/2 + p_2}{p-p_1-p_2+2}.$$

Let $S' \subset E(G), G-S'$ be k-tree, and $\tau' = \tau'(G-S') = p_2, \omega' = \omega'(G-S') = p-p_2+1$. Then

$$|S'| = pk - k(k+1)/2 - [p_2k - k(k+1)/2] = pk - p_2.$$

Thus,

$$sc(S') = \frac{pk - p_2k + p_2}{p-p_2+1},$$

and so

$$\begin{aligned} sc(S) - sc(S') &\geq \frac{(p-p_1-p_2)k + k(k+1)/2 + p_2}{p-p_1-p_2+2} - \frac{pk-p_2k+p_2}{p-p_2+1} \\ &= \frac{(p-p_2+1)[(p-p_1-p_2)k + k(k+1)/2 + p_2] - (p-p_1-p_2+2)(pk+p_2-p_2k)}{(p-p_1-p_2+2)(p-p_2+1)}. \end{aligned}$$

Now, it is easy to show that

$$\begin{aligned} &(p-p_2+1)[(p-p_1-p_2)k + k(k+1)/2 + p_2] - (p-p_1-p_2+2)(pk-p_2k+p_2) \\ &= (p-p_2)k(k-1)/2 + k(k+1)/2 + p_2(p_1-1) - k(p_1-1) - k \\ &= (p-p_2)k(k-1)/2 + k(k-1)/2 + (p_2-k)(p_1-1) > 0. \end{aligned}$$

$sc(S) > sc(S')$, a contradiction.

Case 2 $p_1 \leq k, p_2 \leq k$. Then

$$|E(H_1)| \leq p_1(p_1-1)/2, |E(H_2)| \leq p_2(p_2-1)/2.$$

Thus,

$$|S| \geq [pk - k(k+1)/2] - [p_1(p_1-1)/2 + p_2(p_2-1)].$$

Thus,

$$sc(S) \geq \frac{kp - k(k+1)/2 - [p_1(p_1-1)/2 + p_2(p_2-1)/2] + p_2}{p - p_1 + 2 - p_2}.$$

Let $S' \subset E(G)$ and $G - S'$ be the complete graph with p_2 vertices.

Then

$$sc(S') = \frac{kp - k(k+1)/2 - p_2(p_2-1)/2 + p_2}{p - p_2 + 1},$$

$$sc(S) - sc(S') \geq \frac{kp - k(k+1)/2 - [p_1(p_1-1)/2 + p_2(p_2-1)/2] + p_2}{p - p_1 + 2 - p_2} - \frac{kp - k(k+1)/2 - p_2(p_2-1)/2 + p_2}{p - p_2 + 1}.$$

Now, it is easy to show that

$$\begin{aligned}&(p-p_2+1)[kp - k(k+1)/2 - p_1(p_1-1)/2 - p_2(p_2-1)/2 + p_2] - \\ &\quad (p - p_1 + 2 - p_2)[kp - k(k+1)/2 - p_2(p_2-1)/2 + p_2] \\ &= (p_1 - 1)[pk - k(k+1)/2 - p_2(p_2-1)/2 + p_2 - p_1(p - p_2 + 1)/2] \\ &\geq (p_1 - 1)[pk - k(k+1)/2 - p_2(p_2-1)/2 + p_2 - p_2(p - p_2 + 1)/2] \\ &= (p_1 - 1)[pk/2 - k(k+1)/2 + pk/2 - p_2 p/2 + p] > 0,\end{aligned}$$

Thus, since $p_2 \geq p_1$ and since $p \geq k+1, k \geq p_2$, $sc(S) > sc(S')$, a contradiction.

Case 3 $p_1 \leq k, p_2 \geq k+1$.

We have

$$|E(H_1)| \leq p_1(p_1-1)/2, |E(H_2)| \leq p_2 k - k(k+1)/2.$$

$$|S| \geq pk - k(k+1)/2 - [p_1(p_1-1)/2 + p_2 k - k(k+1)/2] = pk - p_2 k - p_1(p_1-1)/2.$$

Thus,

$$sc(S) \geq \frac{pk - p_2 k - p_1(p_1-1)/2 + p_2}{p - p_1 - p_2 + 2}.$$

Let $S' \subset E(G)$ and $G - S'$ be k-tree with p_2 vertices. Then

$$\begin{aligned} sc(S') &= \frac{pk - k(k+1)/2 - [p_2 k - k(k+1)/2 + p_2}{p - p_2 + 1} \\ &= \frac{pk - p_2 k + p_2}{p - p_2 + 1}.\end{aligned}$$

It is easy to show that

$$\begin{aligned}&(p - p_2 + 1)[pk - p_2 k - p_1(p_1-1)/2 + p_2] - (p - p_1 - p_2 + 2)(pk - p_2 k + p_2) \\ &= (p - p_2)(p_1 - 1)(2k - p_1)/2 + (2p_2 - p_1)(p_1 - 1)/2 > 0,\end{aligned}$$

since $p_1 \leq k, p_2 \geq p_1, p \geq p_2, p_1 \geq 3$. We get $sc(S) > sc(S')$, a contradication.

From three different cases, we have, $G - S$ has at most one nontrivial component.

From Lemmas 6 and 7, we have, G is strictly edge-tenacious. □

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图的边韧性度

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摘 要: 文 [1] 中, 定义图 $G(V,E)$ 的边韧性度定义为 $\min\{\frac{|S|+\tau(G-S)}{\omega(G-S)} : S \subseteq E(G)\}$, 这里, $\tau(G-S)$ 和 $\omega(G-S)$ 分别表示 $G-S$ 中最大分支的顶点数和连通分支数. 这是一个能衡量网络图稳定性较好的参数, 因为它不仅考虑到了图 $G-S$ 的分支数也考虑到了它的阶数. 在以前的工作中, 作者得到了边韧性度图的一个充要条件. 利用这些结果证明了 K- 树是严格边韧性度图, 并找到了边韧性度与较高阶的边坚韧度和边坚韧度之间的关系.

关键词: 边割集; 严格边韧性度图; K- 树; 较高阶的边坚韧度; 边坚韧度.