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On the Jacobi Elliptic Function Expansion Method *

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Abstract: The main idea of this method is to take full advantage of the elliptic equation that Jacobi elliptic functions satisfy and use its solutions to replace Jacobi elliptic functions in Jacobi elliptic function method. Some illustrative equations are investigated by this means.

Key words: Jacobi elliptic function; periodic wave solution; shock wave solution; Wu algebraic elimination.

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1. Introduction

In recent years, directly searching for exact solutions of nonlinear partial differential equations (PDEs) has become more and more attractive partly due to the availability of computer symbolic systems such as Maple and Mathematica, which allow us to perform some complicated and tedious algebraic calculation on a computer as well as help us to find new exact solutions of PDEs. A number of methods have been presented, such as inverse scattering theory[1], Hirotas's bilinear method[2], the truncated Painlevé expansion[3], homogeneous balance method[4], the hyperbolic tangent function series method[5], the sine-cosine method[6]. One of the most effectively straightforward method to construct exact solutions of PDEs is the Jacobi elliptic function method[7,8]. It is an extended tanh function method, since $\mathrm{sn}\xi = \mathrm{sn}(\xi, m) = \tanh\xi$ if $m \to 1$. Let us simply describe the Jacobi elliptic function expansion method. Consider a given PDE, say in two variables

$$H(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{1.1}$$

The fact that the solutions of many nonlinear equations can be expressed as a finite series of Jacobi elliptic functions motivates us to seek the solutions of Eq.(1.1) in the form

$$u(x,t) = \sum_{i=0}^{n} a_i \mathrm{sn}^i(\xi), \tag{1.2}$$

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where $\xi = k(x-ct)$, k and c are the wave number and the wave speed respectively, and n is a positive integer that can be determined by balancing the linear term of highest order with the nonlinear term in Eq.(1.1), and $a_0, a_1, \cdots, a_n$ are parameters to be determined. Substituting (1.2) into Eq.(1.1) yields a set of algebraic equations for $a_0, a_1, \cdots, a_n$ because all coefficients of $\mathrm{sn}^i\xi\mathrm{cn}\xi dn\xi$ have to vanish. From these relations,$a_0, a_1, \cdots, a_n$ can be determined. The purpose of this paper is to present an improved Jacobi elliptic function method.

2. Improved Jacobi elliptic function expansion method

The main idea of our method is to take full advantage of the elliptic equation which the Jacobi elliptic functions satisfy and use its solutions to replace the Jacobi elliptic functions. One of the desired equation reads

$$F'^2 = A + BF^2 + CF^4, \tag{2.1}$$

where $' := \mathrm{d}/\mathrm{d}\xi$, and A, B, C are constants. The solutions of Eq.(1.1) can be expressed in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{n} a_i F^i, \tag{2.2}$$

where $\xi = k(x-ct)$, k and c are the wave number and the wave speed respectively, and n is a positive integer that can be determined by balancing the linear term of highest order with the nonlinear term in Eq.(1.1),and $a_0, a_1, \cdots, a_n$ are parameters to be determined. Here, k and c can be regarded as parameters to be determined, too.

Case 1. If $\begin{cases} A=1 \\ B=-(1+m^2) \\ C=m^2 \end{cases}$, then (2.1) becomes $F'^2 = (1-F^2)(1-m^2F^2)$, which has solution $F = \mathrm{sn}\xi, \mathrm{cd}\xi$.

Case 2. If $\begin{cases} A=1-m^2 \\ B=2m^2-1 \\ C=-m^2 \end{cases}$, then (2.1) becomes $F'^2 = (1-F^2)(m^2F^2+1-m^2)$,which has solution $F = \mathrm{cn}\xi$.

Case 3. If $\begin{cases} A=m^2-1 \\ B=2-m^2 \\ C=-1 \end{cases}$, then (2.1) becomes $F'^2 = (1-F^2)(F^2+m^2-1)$, which has solution $F = \mathrm{dn}\xi$.

Case 4. If $\begin{cases} A=m^2 \\ B=-(1+m^2) \\ C=1 \end{cases}$, then (2.1) becomes $F'^2 = (1-F^2)(m^2-F^2)$, which has solution $F = \mathrm{ns}\xi, \mathrm{dc}\xi$.

Case 5. If $\begin{cases} A=-m^2 \\ B=2m^2-1 \\ C=1-m^2 \end{cases}$, then (2.1) becomes $F'^2 = (1-F^2)(m^2F^2-F^2-m^2)$, which has solution $F = \mathrm{nc}\xi$.

Case 6. If $\begin{cases} A=-1 \\ B=2-m^2 \\ C=m^2-1 \end{cases}$, then (2.1) becomes $F'^2=(1-F^2)(F^2-m^2F^2-1)$, which has solution $F=\text{nd}\xi$.

Case 7. If $\begin{cases} A=1 \\ B=2-m^2 \\ C=1-m^2 \end{cases}$, then (2.1) becomes $F'^2=(1+F^2)(F^2-m^2F^2+1)$, which has solution $F=\text{sc}\xi$.

Case 8. If $\begin{cases} A=1 \\ B=2m^2-1 \\ C=-m^2(1-m^2) \end{cases}$, then (2.1) becomes $F'^2=(1+m^2F^2)(1+m^2F^2-F^2)$, which has solution $F=\text{sd}\xi$.

Case 9. If $\begin{cases} A=1-m^2 \\ B=2-m^2 \\ C=1 \end{cases}$, then (2.1) becomes $F'^2=(1+F^2)(F^2+1-m^2)$, which has solution $F=\text{cs}\xi$.

Case 10. If $\begin{cases} A=-m^2(1-m^2) \\ B=2m^2-1 \\ C=1 \end{cases}$, then (2.1) becomes $F'^2=(m^2+F^2)(F^2+m^2-1)$, which has solution $F=\text{ds}\xi$.

We can see that $\text{sn}\xi=\text{sn}(\xi)=\text{sn}(\xi,m)$ and $\text{cn}\xi$ are special solutions of Eq.(2.1). If $m\to 1$, then $\text{sn}\xi\to\tanh\xi$,$\text{cn}\xi\to\text{sech}\xi$. If $m\to 0$,then $\text{sn}\xi\to\sin\xi$,$\text{cn}\xi\to\cos\xi$. Thus, the tanh function method, sine-cosine method, and other Jacobi elliptic function method are special cases of ours. Our method is described as the following: The nonlinear equation is written as Eq.(1.1). Introducing the similarity variable $\xi=k(x-ct)$, the travelling wave solutions $u(\xi)$ satisfy

$$H'(u,u',u'',\cdots)=0. \tag{2.3}$$

By balancing the highest order of linear term with the nonlinear term in Eq.(2.3), we can determine n in Eq.(2.2).

Substituting (2.2) and (2.1) into Eq.(2.3) yields a set of algebraic equations for $a_0,a_1,\cdots,a_n$ because all coefficients of F^i have to vanish. From these relations, $a_0,a_1,\cdots,a_n$ can be determined. In the following we illustrate our method by considering some important equations.

3. Applications

Example 1 Consider the Gardner equation[9]

$$u_t+uu_x-\alpha u^2u_x+\beta u_{xxx}=0. \tag{3.1}$$

Let $u=u(\xi),\xi=k(x-ct)$, then (3.1) becomes

$$-cu'+uu'-\alpha u^2u'+\beta k^2u'''=0. \tag{3.2}$$

By balancing the highest order of derivative term and nonlinear term in (3.1), we obtain $n=1$. Therefore, we choose the following ansatz

$$u = a_0 + a_1 F \tag{3.3}$$

Taking the derivative of each side of (2.1), we obtain

$$F'' = BF + 2CF^3. \tag{3.31}$$

Taking the derivative of each side of (3.31), we obtain

$$F''' = BF' + 6CF^2F'. \tag{3.32}$$

Substituting (3.3) into (3.2) along with Eq.(3.32) yields a system of equations w.r.t. F^i. Setting the coefficients of F^i in the obtained system of equations to zero, we can deduce the following set of algebraic polynomials with the respect unknowns a_0, a_1, namely:

$$-\frac{\alpha}{3}a_1^3 + 2a_1Ck^2\beta = 0,$$

$$\frac{1}{2}a_1^2 - \alpha a_0 a_1^2 = 0,$$

$$a_0a_1 - \alpha a_0^2 a_1 - a_1 c + a_1 Bk^2\beta = 0, \tag{3.33}$$

for which we find

$$a_0 = \frac{1}{2\alpha}, a_1 = \pm\sqrt{6\beta k^2 C}, c = a_0 - \alpha a_0^2 + Bk^2\beta = \frac{1+4\alpha Bk^2\beta}{4\alpha}. \tag{3.4}$$

Substituting (3.4) into (3.3) gives

$$u(x,t) = \frac{1}{2\alpha} \pm \sqrt{6\beta k^2 C}F. \tag{3.5}$$

Using the special solutions of Eq.(2.1), we obtain the exact periodic solutions of (3.1).

1. $\beta > 0$

$$u_1 = \frac{1}{2\alpha} \pm \sqrt{6\beta}mk\mathrm{sn}[k(x - \frac{1-4\alpha\beta k^2(1+m^2)}{4\alpha}t)], \tag{3.6}$$

$$u_2 = \frac{1}{2\alpha} \pm \sqrt{6\beta}k\mathrm{ns}[k(x - \frac{1-4\alpha\beta k^2(1+m^2)}{4\alpha}t)], \tag{3.7}$$

$$u_3 = \frac{1}{2\alpha} \pm \sqrt{6\beta(1-m^2)}k\mathrm{nc}[k(x - \frac{1+4\alpha\beta k^2(2m^2-1)}{4\alpha}t)], \tag{3.8}$$

$$u_4 = \frac{1}{2\alpha} \pm \sqrt{6\beta(1-m^2)}\mathrm{sc}[k(x - \frac{1+4\alpha\beta k^2(2-m^2)}{4\alpha}t)], \tag{3.9}$$

$$u_5 = \frac{1}{2\alpha} \pm \sqrt{6\beta}k\mathrm{cs}[k(x - \frac{1+4\alpha\beta k^2(2-m^2)}{4\alpha}t)], \tag{4.0}$$

$$u_6 = \frac{1}{2\alpha} \pm \sqrt{6\beta}k\mathrm{ds}[k(x - \frac{1+4\alpha\beta k^2(2m^2-1)}{4\alpha}t)], \tag{4.1}$$

$$u_7 = \frac{1}{2\alpha} \pm \sqrt{6\beta} mk\mathrm{cd}[k(x - \frac{1 - 4\alpha\beta k^2(1+m^2)}{4\alpha}t)], \tag{4.2}$$

$$u_8 = \frac{1}{2\alpha} \pm \sqrt{6\beta} k\mathrm{dc}[k(x - \frac{1 - 4\alpha\beta k^2(1+m^2)}{4\alpha}t]. \tag{4.3}$$

2. $\beta < 0$

$$u_9 = \frac{1}{2\alpha} \pm \sqrt{-6\beta} mk\mathrm{cn}[k(x - \frac{1 + 4\alpha\beta k^2(2m^2-1)}{4\alpha}t)], \tag{4.4}$$

$$u_{10} = \frac{1}{2\alpha} \pm \sqrt{-6\beta} k\mathrm{dn}[k(x - \frac{1 + 4\alpha\beta k^2(2-m^2)}{4\alpha}t)], \tag{4.5}$$

$$u_{11} = \frac{1}{2\alpha} \pm \sqrt{6\beta(m^2-1)} k\mathrm{nd}[k(x - \frac{1 + 4\alpha\beta k^2(2-m^2)}{4\alpha}t)], \tag{4.6}$$

$$u_{12} = \frac{1}{2\alpha} \pm \sqrt{6\beta(m^2-1)} mk\mathrm{sd}[k(x - \frac{1 + 4\alpha\beta k^2(2m^2-1)}{4\alpha}t)], \tag{4.7}$$

If $m \to 1$, then we obtain the shock wave solution of (3.1)

$$u_{13} = \frac{1}{2\alpha} \pm \sqrt{6\beta} k \tanh[k(x - \frac{1 - 8\alpha\beta k^2}{4\alpha}t)]. \tag{4.8}$$

Example 2 Consider the (2+1) dimensional KP equation[10]

$$(u_t + 6uu_x + \beta u_{xxx})_x + \varepsilon u_{yy} = 0. \tag{5.0}$$

Let $u = u(\xi), \xi = k(ax + by - \omega t)$,then (5.0) becomes

$$-a\omega u'' + 6(a^2 u'^2 + a^2 uu'' + \beta k^2 a^4 u^{(4)}) + \varepsilon b^2 u'' = 0. \tag{5.1}$$

By balancing the highest order of derivative term and nonlinear term in (5.1), we obtain $n = 2$. Therefore, we choose the following ansatz

$$u = u(\xi) = a_0 + a_1 F + a_2 F^2. \tag{5.2}$$

Substituting (5.2) into (5.1) along with (3.31) yields a set of algebraic equations for a_0, a_1, a_2

$$60a^2 a_2^2 C + 720a^4 a_2 C^2 k^2 \beta = 0,$$

$$72a^2 a_1 a_2 + 144a^4 a_1 C^2 k^2 \beta = 0,$$

$$48a^2 a_2^2 B + 18a^2 a_1^2 C + 36a^2 a_0 a_2 C - 6aa_2\omega C + 720a^4 a_2 BCk^2\beta + 6a_2 b^2 \varepsilon C = 0,$$

$$54a^2 a_1 a_2 B + 12a^2 a_0 a_1 C - 2a_1 a\omega C + 120a^4 a_1 BCk^2\beta + 2a_1 b^2 C\varepsilon = 0,$$

$$36a^2 Aa_2^2 + 12a^2 a_1^2 B + 24a^2 a_0 a_2 B - 4a_2 a\omega B + 6a^4 a_2(16B^2 + 72AC)k^2\beta + 4a_2 b^2 B\varepsilon = 0,$$

$$36a^2 Aa_1 a_2 + 6a^2 a_0 a_1 B - a_1 a\omega B + 6a^4 a_1(B^2 + 12AC)k^2\beta + a_1 b^2 B\varepsilon = 0,$$

$$6a^2 Aa_1^2 + 12a^2 Aa_0 a_2 - 2Aa_2 a\omega + 48a^4 Aa_2 Bk^2\beta + 2Aa_2 b^2\varepsilon = 0, \tag{5.3}$$

for which, with the aid of Wu algebraic elimination, we find

$$a_1 = 0, a_2 = -12a^2k^2\beta C, a_0 = \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2 B}{6a^2}. \tag{5.4}$$

Substituting (5.4) into (5.2) gives

$$u(x,t) = \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2 B}{6a^2} - 12a^2k^2\beta CF^2. \tag{5.5}$$

Using the special solutions of Eq.(2.1), we obtain the exact periodic solutions of (5.0)

$$\begin{aligned} u_1 &= \frac{a\omega - \varepsilon b^2 + 24a^4\beta k^2(1+m^2)}{6a^2} - 12a^2k^2\beta m^2 \mathrm{sn}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2m^2-1)}{6a^2} + 12a^2k^2\beta m^2 \mathrm{cn}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2-m^2)}{6a^2} + 12a^2k^2\beta \mathrm{dn}^2[k(ax+by-\omega t)], \end{aligned} \tag{5.6}$$

$$\begin{aligned} u_2 &= \frac{a\omega - \varepsilon b^2 + 24a^4\beta k^2(1+m^2)}{6a^2} - 12a^2k^2\beta ns^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2m^2-1)}{6a^2} - 12a^2k^2\beta \mathrm{ds}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2-m^2)}{6a^2} - 12a^2k^2\beta \mathrm{cs}^2[k(ax+by-\omega t)], \end{aligned} \tag{5.7}$$

$$\begin{aligned} u_3 &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2m^2-1)}{6a^2} - 12a^2k^2\beta(1-m^2) \mathrm{nc}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2-m^2)}{6a^2} - 12a^2k^2\beta(1-m^2) \mathrm{sc}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 + 24a^4\beta k^2(1+m^2)}{6a^2} - 12a^2k^2\beta \mathrm{dc}^2[k(ax+by-\omega t)]. \end{aligned} \tag{5.8}$$

$$\begin{aligned} u_4 &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2-m^2)}{6a^2} - 12a^2k^2\beta(m^2-1) \mathrm{nd}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2(2m^2-1)}{6a^2} - 12a^2k^2\beta m^2(m^2-1) \mathrm{sd}^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 + 24a^4\beta k^2(1+m^2)}{6a^2} - 12a^2k^2\beta m^2 \mathrm{cd}^2[k(ax+by-\omega t)]. \end{aligned} \tag{5.9}$$

If $m \to 1$, then we obtain the solitary wave solutions of (5.0)

$$\begin{aligned} u_5 &= \frac{a\omega - \varepsilon b^2 + 48a^4\beta k^2}{6a^2} - 12a^2k^2\beta \tanh^2[k(ax+by-\omega t)] \\ &= \frac{a\omega - \varepsilon b^2 - 24a^4\beta k^2}{6a^2} + 12a^2k^2\beta \mathrm{sech}^2[k(ax+by-\omega t)]. \end{aligned} \tag{6.0}$$

4. Conclusion

We have presented an improved Jacobi elliptic function expansion method and used it to solve some important nonlinear PDEs.In fact, this method is readily applicable to a large variety of nonlinear PDEs. In contrast to other Jacobi elliptic function expansion method, some additional merits are available for our method.

First, all the nonlinear PDEs which can be solved by other Jacobi elliptic function expansion method can be solved easily by our method. Second, we used only the special solutions of (2.1). If we use other solutions of (2.1), we can obtain more wave solutions. Third, our method contains not only the hyperbolic tangent expansion method, but also the method used in [7-9]. Thus, our method is a generalized method for abundant nonlinear PDEs. Many other nonlinear PDEs can be solved by our method.

References:

[1] GARDNER C S, GREENE J M, KRUSKAL M D, et al. *Method for solving the KdV equation* [J]. Phys. Rev. Lett., 1967, **19**: 1095–1097.

[2] HU Xing-biao, MA Wen-xiu. *Application of Hirota's bilinear formalism to the Toeplitz lattice-some special soliton-like solutions* [J]. Phys. Lett. A, 2002, **293**: 161–165.

[3] WEIS J, TABOR M, GARNEVALE G. *The painlevé property for partial differential equations* [J]. J. Math. Phys., 1983, **24**: 522–526.

[4] WANG Ming-liang, ZHOU Yu-bin, LI Zhi-bin. *Applications of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics* [J]. Phys. Lett. A, 1996, **216**: 67–75.

[5] TIBOR B, BÉLA L, CSABA M, et al. *The hyperbolic tangent distribution family* [J]. Powder Technology, 1998, **97**: 100–108.

[6] YAN Chun-tao, *A simple transformation for nonlinear waves* [J]. Phys. Lett. A, 1996, **224**: 77–84.

[7] LIU Shi-kuo, FU Zun-tao, LIU Shi-da, et al. *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations* [J]. Phys. Lett. A, 2001, **289**: 69–74.

[8] ZHOU Yu-bin, WANG Ming-liang, WANG Yue-ming. *Periodic wave solutions to a coupled KdV equations with variable coefficients* [J]. Phys. Lett. A, 2003, **308**: 31–36.

[9] LIU Shi-da, LIU Shi-kuo. *Exact travelling wave solutions of nonlinear evolution equations* [J]. Mathematics in Practice and Theory, 1998, **28**: 289–301.

[10] BERNARD D, HARVEY S. *The KP equation with quasiperiodic initial* data [J]. Physica D, 1998, **123**: 123–152.

关于 Jacobi 椭圆函数展开法

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摘 要: 主要利用 Jacobi 椭圆函数所满足的方程并用其解代替 Jacobi 椭圆函数以求非线性偏微分方程的周期解，并举例说明该方法的应用.

关键词: Jacobi 椭圆函数; 周期波解; 冲击波解; 吴代数消元法.