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Strong Converse Inequality for Modified Szász Operators *

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Abstract: In this paper, we show the direct and converse theorems of strong type of approximation for modified Szász operators. From these theorems, the characterization of approximation for these operators is derived. The obtained results are similar to the corresponding ones of the Szász operators.

Key words: Modified Szász operators; strong converse inequality; approximation.

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1. Introduction

Let $C[0,\infty)$ be the set of continuous and bounded function on $[0,\infty)$, $P_{n,k}(x) = e^{-nx}(nx)^k/k!, x \geq 0, n \in N$. We denote the Szász operators by

$$S_n f = S_n(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x) f(\frac{k}{n}).$$

In [1], S. M. Mazhar and V. Totik gave a modification for the Szász operators:

$$M_n f = M_n(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x)\, \Phi_{n,k}(f), \tag{1}$$

where

$$\Phi_{n,k}(f) = \begin{cases} f(0), & k=0; \\ n\int_0^\infty P_{n,k-1}(t) f(t)\,\mathrm{d}t, & k=1,2,\cdots. \end{cases}$$

For $\varphi(x) = \sqrt{x}$ and $f \in C[0,\infty)$, the Ditzian–Totik modulus of r order is given by (see also [2])

$$\omega_\varphi^r(f,t) = \sup_{0<h\leq t} \|\Delta_{h\varphi}^r f\|,$$

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where

$$\Delta^r_{h\varphi} f(x) = \begin{cases} \sum_{i=0}^{r} (-1)^i C_r^i f(x + (r/2 - i)h\varphi(x)), & x - rh\varphi(x)/2 \in [0, \infty); \\ 0, & \text{otherwise.} \end{cases}$$

The K-functional used in the present paper is defined by

$$K^r_\varphi(f, t^r) = \inf_{g \in D_r} \{\|f - g\| + t^r \|\varphi^r g^{(r)}\|\},$$

where $D_r = \{g \in C[0, \infty) : g^{(r-1)} \in \text{A.C.loc.}, \varphi^r g^{(r)} \in C[0, \infty)\}$ is a Sobolev space. It was shown in [2] that

$$C^{-1}\omega^r_\varphi(f, t) \le K^r_\varphi(f, t^r) \le C\omega^r_\varphi(f, t). \tag{2}$$

Here and in the following, C is a positive constant independent of n, f and x, and its value may be different at different occurrence.

In this note, we discuss the problem of global approximation by $M_n f$. We estimate the upper and lower bounds of the rate of approximation by using the Ditzian–Totik modulus of smoothness. A strong converse inequality, which is an analogue with corresponding result for the Szász operators [3], is obtained.

Our main results are given by the following theorems.

Theorem 1 *If $f \in C[0, \infty)$, then*

$$\|M_n f - f\| \le C\omega^2_\varphi(f, 1/\sqrt{n}).$$

Theorem 2 *If $f \in C[0, \infty)$, then there exists an integer $m \ge 17n$, such that*

$$\omega^2_\varphi(f, 1/\sqrt{n}) \le C(\|M_n(f) - f\| + \|M_m(f) - f\|).$$

From Theorems 1 and 2, it is easy to derive that

Corollary 1 *Under the condition of Theorem 2, we have*

$$\omega^2_\varphi(f, 1/\sqrt{n}) \le C(\|M_n(f) - f\| + \|M_m(f) - f\|) \le C\omega^2_\varphi(f, 1/\sqrt{n}).$$

Corollary 2 *For $f \in C[0, \infty)$, $0 < \alpha \le 1$, we have*

$$\|M_n f - f\| = O(n^{-\alpha})$$

if and only if $\omega^2_\varphi(f, t) = O(t^{2\alpha})$.

2. Some lemmas

To prove Theorems 1 and 2, we give some lemmas in this part.

Lemma 1 *For $f \in C^r[0,\infty), r \in N$, we have*

$$M_n^{(r)}(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x)\, n \int_0^{\infty} f^{(r)}(t)\, P_{n,k+r-1}(t)\, dt.$$

Proof For convenience we suppose that $P_{n,k}(x) \equiv 0$ for $k < 0$. Since

$$P'_{n,k}(x) = n\left(P_{n,k-1}(x) - P_{n,k}(x)\right)$$

and

$$P''_{n,k}(x) = n^2\left(P_{n,k}(x) - 2P_{n,k-1}(x) + P_{n,k-2}(x)\right),$$

we have from the induction $P_{n,k}^{(r)}(x) = n^r \sum_{j=0}^{r} C_r^j (-1)^j P_{n,k-r+j}(x)$, which implies that

$$\begin{aligned}
M_n^{(r)}(f,x) &= n^r \sum_{j=0}^{r} C_r^j (-1)^j \sum_{k=0}^{\infty} \Phi_{n,k}(f)\, P_{n,k-r+j}(x) \\
&= n^r \sum_{j=0}^{r} C_r^j (-1)^j \sum_{k=0}^{\infty} \Phi_{n,k+r-j}(f)\, P_{n,k}(x) \\
&= n^r \sum_{k=0}^{\infty} P_{n,k}(x) \sum_{j=0}^{r} (-1)^{r-j} C_r^j \Phi_{n,k+j}(f) \\
&= n^r \sum_{k=0}^{\infty} (-1)^r P_{n,k}(x)\, \delta_{nk}^{(r)}(f),
\end{aligned}$$

where $\delta_{nk}^{(r)}(f) = \sum_{j=0}^{r} C_r^j (-1)^j \Phi_{n,k+j}(f)$. We need to prove for $k = 0, 1, 2, \cdots$

$$\delta_{nk}^{(r)}(f) = (-1)^r n^{1-r} \int_0^{\infty} f^{(r)}(t)\, P_{n,k+r-1}(t)\, dt. \tag{3}$$

In fact, if $k = 0$, then

$$\begin{aligned}
\delta_{nk}^{(r)}(f) &= f(0) + \sum_{j=1}^{r} C_r^j (-1)^j \Phi_{n,j}(f) \\
&= \sum_{j=1}^{r} C_r^j (-1)^j n \int_0^{\infty} \left(f(t) - f(0)\right) P_{n,j-1}(t)\, dt \\
&= n^{1-r} \int_0^{\infty} \left(f(t) - f(0)\right) P_{n,r-1}^{(r)}(t)\, dt \\
&= (-1)^r n^{1-r} \int_0^{\infty} f^{(r)}(t)\, P_{n,r-1}(t)\, dt.
\end{aligned}$$

Similarly, we can prove that (3) is valid for the case $k \ge 1$. The proof of Lemma 1 is complete.

Lemma 2 *For $M_n f$ given by (1), there hold*

$$\|M_n f\| \le \|f\|, \quad f \in C[0,\infty),$$

$$\left\|\varphi^2 M_n f\right\| \le 2n\|f\|, \quad f \in C[0,\infty),$$

$$\left\|\varphi^2 M_n'' f\right\| \le \left\|\varphi^2 f''\right\|, \quad f \in D_2,$$

and

$$\|M_n f - f\| \le \frac{2}{n}\|\varphi^2 f''\|, \quad f \in D_2.$$

Proof From the fact that $|\Phi_{n,k}(f)| \le \|f\|$ and Lemma 1 , we can prove the above inequalities by using the standard methods (see also [2], [3] and [4]). We omit the details.

Lemma 3 *For $f \in D_2$, we have*

$$\|\varphi^3 M_n^{(3)} f\| \le 2n^{1/2}\|\varphi^2 f''\|.$$

Proof From Lemma 1, it is easy to follow that

$$\begin{aligned}
|\varphi^3(x) M_n^{(3)}(f,x)| &= \left| n\sum_{k=0}^{\infty} \varphi^3(x) P'_{n,k}(x) \int_0^{\infty} f''(t) P_{n,k+1}(t)\,\mathrm{d}t \right| \\
&= \left| \sum_{k=0}^{\infty} \varphi(x) \left(\frac{k}{n} - x\right) P_{n,k}(x) \frac{n^3}{k+1} \int_0^{\infty} \varphi^2(t) f''(t) P_{n,k}(t)\,\mathrm{d}t \right| \\
&\le \left\|\varphi^2 f''\right\| n^2 \sum_{k=0}^{\infty} \varphi(x) \frac{1}{k+1} \left|\frac{k}{n} - x\right| P_{n,k}(x).
\end{aligned}$$

Thus, we have for $0 \le x \le 1/n$

$$\begin{aligned}
\left|\varphi^3(x) M_n^{(3)}(f,x)\right| &\le \left\|\varphi^2 f''\right\| n^2 x^{1/2} \left(\sum_{k=0}^{\infty} (k/n - x)^2 P_{n,k}(x)\right)^{1/2} \\
&\le \left\|\varphi^2 f''\right\| n^2 x^{1/2} \left(\frac{x}{n}\right)^{1/2} \le n^{1/2} \left\|\varphi^2 f''\right\|.
\end{aligned}$$

For $x > 1/n$, we see

$$\begin{aligned}
\left|\varphi^3(x) M_n^{(3)}(f,x)\right| &\le \left\|\varphi^2 f''\right\| \frac{n}{\sqrt{x}} \sum_{k=0}^{\infty} |k/n - x| P_{n,k+1}(x) \\
&\le \left\|\varphi^2 f''\right\| \frac{n}{\sqrt{x}} \left(\sum_{k=0}^{\infty} \left|\frac{k+1}{n} - x\right| P_{n,k+1}(x) + \frac{1}{n}\right) \\
&\le \left\|\varphi^2 f''\right\| \frac{n}{\sqrt{x}} \left(\left(\frac{x}{n}\right)^{1/2} + \frac{1}{n}\right) \\
&\le 2n^{1/2} \left\|\varphi^2 f''\right\|.
\end{aligned}$$

Hence, the proof of Lemma 3 is complete.

Lemma 4 *For $f \in D_2$ and $\varphi^3 f^{(3)} \in C[0,\infty)$, we have*

$$\left\|M_n f - f - \frac{1}{n}\varphi^2 f''\right\| \le 32n^{-3/2}\left\|\varphi^3 f^{(3)}\right\|.$$

Proof Using Taylor formula

$$f(u) = f(x) + (u-x) f'(x) + \frac{1}{2}(u-x)^2 f''(x) + \frac{1}{2}\int_x^u (u-v)^2 f^{(3)}(v)\,\mathrm{d}v,$$

and the facts $M_n(f,1)=1, M_n(f,t-x)=0$ and $M_n\left((t-x)^2,x\right)=\frac{2x}{n}$, we have

$$M_n(f,x) - f(x) - \frac{1}{n}\varphi^2(x) f''(x) = \frac{1}{2}M_n\left(\int_x^u (u-v)^2 f^{(3)}(v)\,\mathrm{d}v, x\right).$$

We only need to prove that

$$A_n(x) = \frac{1}{2}M_n\left(\left|\int_x^u (u-v)^2 \varphi^{-3}(v)\,\mathrm{d}v\right|, x\right) \le 32n^{-3/2}.$$

For $x \ge 1/n$, by the fact $\frac{|u-v|}{v} \le \frac{|u-x|}{x}$ for v being between x and u, we find

$$\begin{aligned}
A_n(x) &\le \frac{1}{2}M_n\left(\frac{|u-x|^{3/2}}{\varphi^3(x)}\left|\int_x^u |u-v|^{1/2}\,dv\right|, x\right)\\
&\le \frac{1}{3}\varphi^{-3}(x) M_n\left(|u-x|^3, x\right)\\
&\le \frac{1}{3}\varphi^{-3}(x)\left(M_n\left((u-x)^2,x\right)\right)^{1/2}\left(M_n\left((u-x)^4,x\right)\right)^{1/2}\\
&= \frac{1}{3}\varphi^{-3}(x)\left(\frac{2x}{n}\cdot\frac{6x}{n^2}\left(\frac{4}{n}+2x\right)\right)^{1/2} \le \sqrt{8}n^{-3/2},
\end{aligned}$$

where the Schwarz inequality and the fact that

$$M_n\left((t-x)^2,x\right) = \frac{2x}{n},\quad M_n\left((t-x)^4,x\right) = \frac{6x}{n^2}\left(\frac{4}{n}+2x\right)$$

are used.

For $0 \le x < 1/n$, we distinguish three cases.

Case 1: If $k=0$, then

$$\begin{aligned}
A_n(x) &= \frac{1}{2}P_{n,0}(x)\left|\Phi_{n,0}\left(\int_x^u (u-v)^2\varphi^{-3}(v)\,\mathrm{d}v\right)\right|\\
&= \frac{1}{2}P_{n,0}(x)\int_0^x v^{1/2}\mathrm{d}v \le \frac{1}{3}n^{-3/2}.
\end{aligned}$$

Case 2: If $k=1$, then

$$\begin{aligned}
A_n(x) &= \frac{1}{2}P_{n,1}(x)\,n\left|\int_0^\infty P_{n,0}(u)\int_x^u (u-v)^2\varphi^{-3}(v)\,\mathrm{d}v\mathrm{d}u\right| \\
&\le \frac{1}{2}P_{n,1}(x)\,n\int_0^\infty P_{n,0}(u)(u-x)^2\left|2\left(1/\sqrt{x}-1/\sqrt{u}\right)\right|\mathrm{d}u \\
&= n^2\sqrt{x}e^{-nx}\left(\left(\int_0^x+\int_x^\infty\right)e^{-nu}(u-x)^2\left|1-\frac{\sqrt{x}}{\sqrt{u}}\right|\mathrm{d}u\right) \\
&\le n^2\sqrt{x}\left(\sqrt{x}\int_0^x e^{-nu}u^{-1/2}(u-x)^2\,\mathrm{d}u+\int_x^\infty e^{-nu}(u-x)^2+\mathrm{d}u\right) \\
&= K_1+K_2.
\end{aligned}$$

We now estimate K_1 and K_2. Since

$$\begin{aligned}
K_1 &\le n^2x\int_0^x e^{-nu}\left(u^{3/2}+x^2u^{-1/2}\right)\mathrm{d}u \\
&\le n^2x\left(x^{5/2}+2e^{-nx}x^{5/2}+2x^2n\int_0^x e^{-nu}\mathrm{d}u^{1/2}\right)\le 5n^{-3/2},
\end{aligned}$$

and

$$\begin{aligned}
K_2 &\le n^2x^{1/2}\int_x^\infty e^{-nx}\left(u^2+x^2\right)\mathrm{d}u \\
&\le n^2x^{1/2}\int_0^\infty\left(\frac{2}{n^2}P_{n,2}(u)+x^2P_{n,0}(u)\right)\mathrm{d}u\le 3n^{-3/2},
\end{aligned}$$

so, it is true for the case 2.

Case 3: If $k\ge 2$, then

$$\begin{aligned}
A_n(x) \le&\frac{1}{3}\sum_{k=2}^\infty P_{n,k}(x)\varphi^{-3}(x)\,n\int_0^\infty P_{n,k-1}(u)\,|u-x|^3\,\mathrm{d}u \\
=&\frac{1}{3}\sum_{k=2}^\infty P_{n,k}(x)\varphi^{-3}(x)\,n\left(\int_{|u-x|\le\frac{k}{n}}+\int_{|u-x|>\frac{k}{n}}\right)P_{n,k-1}(u)\,|u-x|^3\,\mathrm{d}u \\
\le&\frac{1}{3}\sum_{k=2}^\infty P_{n,k}(x)\varphi^{-3}(x)\left(\frac{k}{n}\right)^3+ \\
&\frac{1}{3}\sum_{k=2}^\infty P_{n,k}(x)\frac{n^2}{k}\varphi^{-3}(x)\int_0^\infty P_{n,k-1}(u)(u-x)^4\,\mathrm{d}u \\
=&I_1+I_2.
\end{aligned}$$

We first estimate I_1

$$\begin{aligned}
I_1 &= \frac{1}{3}n^2x^{1/2}\sum_{k=2}^\infty P_{n,k-2}(x)\frac{1}{k(k-1)}\left(\frac{k}{n}\right)^3 \\
&= \frac{1}{3}\sqrt{x}\sum_{k=0}^\infty P_{n,k}(x)\left(\frac{k}{n}+\frac{2}{n}\right)\frac{k+2}{k+1}\le\frac{2}{3}\sqrt{x}\left(x+\frac{2}{n}\right)\le 2n^{-3/2}.
\end{aligned}$$

Next, we estimate I_2. Observing that

$$\begin{aligned} I_2 &\le \frac{n^2}{3}\sum_{k=2}^{\infty}P_{n,k}(x)\,\varphi^{-3}(x)\,k^{-2}\int_0^{\infty}\left(u^4+6u^2x^2+x^4\right)P_{n,k-1}(u)\,\mathrm{d}u \\ &= J_1+J_2+J_3, \end{aligned}$$

and

$$\begin{aligned} J_1 &= \frac{1}{3}n^2\sqrt{x}\sum_{k=2}^{\infty}\frac{(k+1)(k+2)(k+3)}{nk(k-1)}P_{n,k-2}(x) \\ &\le \frac{10}{3}n^2\sqrt{x}\sum_{k=0}^{\infty}\frac{k+3}{n}P_{n,k}(x) \le \frac{40}{3}n^{-3/2}, \\ J_2 &= 2n^2\sqrt{x}\sum_{k=2}^{\infty}P_{n,k}(x)\,\frac{1}{k^2}\int_0^{\infty}u^2P_{n,k-1}(u)\,\mathrm{d}u \\ &= 2\sqrt{x}\sum_{k=2}^{\infty}P_{n,k}(x)\,\frac{k+1}{n} \le 4n^{-3/2}, \\ J_3 &= \frac{1}{3}x^{3/2}\sum_{k=2}^{\infty}\frac{k+1}{k}P_{n,k+1}(x) \le \frac{2}{3}n^{-3/2}, \end{aligned}$$

we obtain $I_2 \le 18n^{-3/2}$. Therefore, we have $A_n(x) \le 32n^{-3/2}$, and the proof of Lemma 4 is complete.

3. Proof of the Theorems

Now, we prove Theorems 1 and 2. From Lemma 2 and (1), we can prove Theorem 1 by the standard method(see [2] and [4]). We omit the details . We use some ideas of [3] and [5] to prove Theorem 2. Let $M_n^2f = M_nM_nf$ be the iterative operators of M_nf. By Lemma 4, Lemma 2 and Lemma 3 we have

$$\begin{aligned} &\left\|M_lM_n^2f + M_n^2f - \frac{1}{n}\varphi^2\left(M_n^2f\right)^{(2)}\right\| \le 32l^{-3/2}\left\|\varphi^3\left(M_n^2f\right)^{(3)}\right\| \\ &\le 32l^{-3/2}\left(2n^{1/2}\left\|\varphi^2\left(M_nf\right)^{(2)}\right\|\right) \\ &\le 64l^{-3/2}n^{1/2}\left(\left\|\varphi^2\left(M_nf-M_n^2f\right)^{(2)}\right\| + \left\|\varphi^2\left(M_n^2f\right)^{(2)}\right\|\right) \\ &\le 64l^{-3/2}n^{1/2}\left(2n\left\|M_nf-f\right\| + \left\|\varphi^2\left(M_n^2f\right)^{(2)}\right\|\right), \end{aligned}$$

Putting $l \ge 17n$, we derive

$$64l^{-3/2}n^{1/2} \le \frac{64}{17\sqrt{17}}\frac{1}{n},$$

which implies

$$\left(1-\frac{64}{17\sqrt{17}}\right)\frac{1}{n}\left\|\varphi^2M_n^2f\right\| \le 128l^{-3/2}n^{3/2}\left\|M_nf-f\right\| + \left\|M_lM_n^2f - M_n^2f\right\|.$$

So

$$\frac{1}{n}\left\|\varphi^2 M_n^2 f\right\| \le C\left(\left(\frac{n}{l}\right)^{3/2}\|M_n f - f\| + \left\|M_l M_n^2 f - M_n^2 f\right\|\right).$$

Recalling (2) and

$$\begin{aligned}\left\|M_l M_n^2 f - M_n^2 f\right\| \le& \left\|M_l M_n^2 f - M_l M_n f\right\| + \|M_l M_n f - M_l f\| + \\ & \|M_l f - f\| + \|M_n f - f\| + \left\|M_n f - M_n^2 f\right\| \\ \le& 4\|M_n f - f\| + \|M_l f - f\|,\end{aligned}$$

we thus have

$$\begin{aligned}\omega_\varphi^2(f, 1/\sqrt{n}) \le CK_\varphi^2(f, 1/n) \le& C\left(\left\|f - M_n^2 f\right\| + \frac{1}{n}\left\|\varphi^2\left(M_n^2 f\right)^{(2)}\right\|\right) \\ \le& C(\|M_n f - f\| + \|M_l f - f\|).\end{aligned}$$

This completes the proof.

References:

[1] MAZHAR S M, TOTIK V. *Approximation by modified Szász operators* [J]. Acta Sci. Math., 1985, **49**: 257–269.

[2] DITZIAN Z, TOTIK V. *Moduli of Smoothness* [M]. Springer-Verlag, Berlin, New York, 1987.

[3] TOTIK V. *Strong converse inequalities* [J]. J. Approx. Theory, 1994, **76**: 369–375.

[4] TOTIK V. *An interpolation theorem and applications to positive operators* [J]. Pacific J. Math., 1984, **111**(2): 447–481.

[5] DITZIAN Z, IVANOV K G. *Strong converse inequalities* [J]. J. Anal. Math., 1993, **61**: 61–111.

修正的 Szász 算子的强逆不等式

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摘 要: 本文证明修正的 Szász 算子逼近的强型正定理和逆定理，从而得到该算子逼近特征的刻画. 所获结果类似于 Szász 算子相应的结果.

关键词: 修正的 Szász 算子；强逆不等式；逼近.