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A Survey of Various Refinements and Generalizations of Hilbert's Inequalities

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Abstract: Expounded in this survey article is a series of refinements and generalizations of Hilbert's inequalities mostly published during the years 1990 through 2002.Those inequalities concerned may be classified into several types (discrete and integral etc.), and various related results obtained respectively by L. C. Hsu, M. Z. Gao, B. C. Yang, J. C. Kuang, Hu Ke and H. Hong et.al are described a little more precisely. Moreover, earlier and recent extensions of Hilbert-type inequalities are also stated for reference. And the new trend and the research ways are also brought forward.

Key words: Hilbert's inequalities; Hardy-Hilbert's inequalities; weight function; Gram matrix; double series.
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1. Introduction

The inequalities of the form

$$\sum_{m,n=1}^{\infty}\frac{a_m b_n}{m+n} \le \pi\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2} \tag{1}$$

and

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{m+n+1} \le \pi\left(\sum_{n=0}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=0}^{\infty}b_n^2\right)^{1/2} \tag{2}$$

are called Hilbert's inequalities for double series[1], where the same constant π in (1) and (2) is known to be best possible, and the equalities in these inequalities hold if and only if $\{a_n\}$or $\{b_n\}$ is a zero-sequence. Subsequently, Hardy extended these results. To be specific, he established the following inequalities[1]:

$$\sum_{m,n=1}^{\infty}\frac{a_m b_n}{m+n} \le \frac{\pi}{\sin\ \pi/p}\left(\sum_{n=1}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q} \tag{3}$$

and

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{m+n+1} \le \frac{\pi}{\sin\ \pi/p}\left(\sum_{n=0}^{\infty}a_n^p\right)^{1/p}\left(\sum_{n=0}^{\infty}b_n^q\right)^{1/q}, \tag{4}$$

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where a_n, $b_n \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq q > 1$. The inequalities (3) and (4) are called Hardy-Hilbert's inequalities for double series. And the equalities in (3) and (4) hold if and only if $\{a_n\}$or $\{b_n\}$ is a zero-sequence.

The above inequalities possess the corresponding forms of the integral, namely

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left\{\int_0^\infty f^2(x)\,dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)\,dx\right\}^{1/2} \tag{5}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin \pi/p} \left\{\int_0^\infty f^p(x)\,dx\right\}^{1/p} \left\{\int_0^\infty g^q(x)\,dx\right\}^{1/q}. \tag{6}$$

In view of the importance of the results in theory and applications, it has been absorbing much interest of analysts. Recently, various improvements and extensions or generalizations appear in a great deal of papers (see the references of the present paper). Taking one with another, they possess mainly the following characteristics:

1) Certain weight functions are introduced, so that a lot of graceful refinements of the above-mentioned inequalities are obtained.

2) Certain parameters are introduced, so that various nice generalizations and extensions of these inequalities are established.

3) Some new inequalities proven have been utilized, so that many new results are yielded.

In the present paper, we shall briefly look back the research status of these inequalities. For each part of the following, we state firstly the discrete form of them and then introduce their integral form.

2. Refinements by choosing weight functions

For years, Hilbert's inequalities have been studied by mathematicians, and thereby various nice results have been obtained. In this section, we look back only the basic status since more ten years. We introduce firstly the results acquired on Hilbert's original inequalities, and then show some achievements on Hardy- Hilbert's inequalities.

In 1990, L. C. Hsu et al. analyzed carefully the original technique of Hardy and introduced the weight function of the form $\omega(n) = \pi - \sqrt{n}/(n+1)$[2] such that the inequality (1) is improved into the following

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{m+n} \leq \left(\sum_{n=1}^\infty \omega(n)\, a_n^2\right)^{1/2} \left(\sum_{n=1}^\infty \omega(n)\, b_n^2\right)^{1/2}. \tag{7}$$

Before long, the weight function $\omega(n)$ was further refined into $\pi - \theta/\sqrt{n}$ by Hsu and Wang[3], where $\theta = 3/\sqrt{2} - 1$ =1.121320343$^+$, and the problem of seeking the greatest possible value of θ for the inequality (7) keeps valid is mentioned. Later on, L.C. Hsu and M.Z. Gao employed different methods respectively to attain an infimum of θ, i.e. $\theta = 1.281^+$[4,5]. And then Gao showed that a supremum of θ is λ $(\lambda = 1.4603545^+)$[6]. As a result, the problem mentioned in [3] is solved.

As for the inequality (2), Gao established an improvement of it in the paper[7]:

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{m+n+1}\le\left\{\sum_{n=0}^{\infty}\omega(n)a_n^2\right\}^{1/2}\left\{\sum_{n=0}^{\infty}\omega(n)b_n^2\right\}^{1/2},$$

where $\omega(n)=\pi-\theta(n)/\sqrt{2n+1}$ with $\theta(n)>0$ $(n=0,1,2,\cdots)$. And then Gao[8] applied Euler-Maclaurin summation formula to estimate the weight function ω as follows:

$$\omega(n)\le\pi-\theta/\sqrt{2n+1}$$

where $\theta=17/20$.

For the Hilbert integral inequality

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y+1}\mathrm{d}x\mathrm{d}y\le\pi\left\{\int_0^{\infty}f^2(x)\,\mathrm{d}x\right\}^{1/2}\left\{\int_0^{\infty}g^2(x)\,\mathrm{d}x\right\}^{1/2}\tag{8}$$

Gao[7] established also the new inequality with weights:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y+1}\le\left\{\int_0^{\infty}\omega(x)f^2(x)\,\mathrm{d}x\right\}^{1/2}\left\{\int_0^{\infty}\omega(x)g^2(x)\,\mathrm{d}x\right\}^{1/2},\tag{9}$$

where $\omega(x)=\pi-2arctg\left(1/\sqrt{2x+1}\right)$. Two years later, Yang[9] built also the following inequality:

$$\omega(x)\le\pi-(\pi-2)\big/(x+1)^{1/2}.\tag{10}$$

And proved that $(\pi-2)$ is the best possible value which the inequality (9) keeps valid.

In a similar way, some new results on Hardy-Hilbert's inequalities were obtained.

In studying of Hardy-Hilbert's inequality (3), the summation with parameter n is estimated, i.e.:

$$\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/x}\le\omega_n(x).\quad x=p,\ q.$$

In 1990, Hsu and Guo[2] took the lead in introducing the following weight function:

$$\omega_n(x)=\frac{\pi}{\sin\,\pi/p}-\frac{n^{1/x}}{(n+1)(x-1)}.$$

The inequality (3) was refined as

$$\sum_{m,n=1}^{\infty}\frac{a_m b_n}{m+n}\le\left\{\sum_{n=1}^{\infty}\omega_n(q)a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\omega_n(p)b_n^q\right\}^{1/q}.\tag{11}$$

Afterwards, the second term of the weight function ω_n was further improved as $n^{1/x}/n(x-1)$ by Hsu[10] and Gao[11]. After two years, Gao[12] gave again the accurate expression of the weight function:

$$\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/r}=\frac{\pi}{\sin\,\pi/p}-\frac{\theta_r(n)}{n^{1-1/r}},$$

where $\theta_r(n) > 0$. $r = p,\ q$.

Before long, Yang and Gao[13,14] got an infimum of $\theta_r(n)$, namely, a fine result on the weight function is obtained:

$$\omega_n(n) = \frac{\pi}{\sin\ \pi/p} - \frac{1-c}{n^{1-1/r}},$$

where c is Euler constant, and $(1-c)$ is proved to be the best constant which the inequality (11) keeps valid. By and by Gao[6] proved that a supremum of $\theta_r(n)$is that

$$\theta_r = \sup_{n\in N}\{\theta_r(n)\} = \frac{r}{r-1} + \frac{1}{2} + \int_1^{\infty} \rho(t)\left(1\Big/rt^{1+1/r}\right)\mathrm{d}t,$$

where $\rho(t) = t - [t] - \frac{1}{2}$. And θ_r is estimated as

$$\frac{r}{r-1} - \frac{1}{2} - \frac{1}{8r} < \theta_r < \frac{r}{r-1} - \frac{1}{2} - \frac{1}{12r}\left(\frac{2}{3}\right)^{1+\frac{1}{r}}.$$

It shows that the inequalities (7) and (11) keep no longer valid, if $\theta_r(n) > \theta_r$.Thus the problems posed in the papers[3],[10] were extended and solved entirely.

Concerning the inequality (4), Yang[15] attained the following nice result:

$$\sum_{m,n=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty} \omega(n,\ p)\, a_n^p\right\}^{1/p} \left\{\sum_{n=0}^{\infty} \omega(n,\ q)\, b_n^q\right\}^{1/q}, \tag{13}$$

where $\omega(n,\ r) = \frac{\pi}{\sin\ \pi/p} - \frac{\ln 2 - c}{(2n+1)^{2-1/r}}$, $r = p,\ q$ and c is Euler constant.

In 1998, Yang and Debnath[16] gave other form on Hardy-Hilbert's inequality with weights:

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \le \left\{\sum_{n=1}^{\infty} \omega(n,\ q)\, a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} \omega(n,\ p)\, b_n^q\right\}^{1/q},$$

where $\omega(n,\ r) = \frac{\pi}{\sin\ \pi/p} - \frac{1}{2n^{1-1/r} + n^{-1/r}}$, $r = p,\ q$.

Besides the above-mentioned, Yang[17] had the following result:

$$\sum_{m,n=0}^{\infty} \frac{a_m b_n}{m+n+1} \le \left\{\sum_{n=0}^{\infty} s(n,\ p)\, a_n^p\right\}^{1/p} \left\{\sum_{n=0}^{\infty} s(n,\ q)\, b_n^q\right\}^{1/q},$$

where $s(n,\ r) = \frac{\pi}{\sin\ \pi/p} - \frac{1}{13(n+1)(2n+1)^{1/r}}$.

If $s(n,\ r)$in the above expression is replaced by $\frac{\pi}{\sin\ \pi/p} - \frac{1}{10(2n+1)^{1+1/r}}$, then the result of the paper[18] follows as a consequence.

3. Generalization by introducing proper parameters

The parameters are properly introduced such that the objects studied are generalized. This is a method employed usually. In this section, we shall summarize various generalizations on Hilbert's inequalities with parameters. First of all, we introduce some new results on discrete form of Hilbert's inequalities, and then state similar ones on integral form of them.

Recently, for discrete form of Hilbert's inequality Yang introduced firstly parameters A, B and λ such that the inequality (1) is generalized. He established the following new inequality in paper[19]:

$$\sum_{m,n=1}^{\infty}\frac{a_m b_n}{(Am+Bn)^{\lambda}}<\frac{1}{(AB)^{\lambda/2}}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right)^{1/2}, \tag{14}$$

where $A,B>0$ and $0<\lambda\le 2, B(p,q)$ is the beta function and the constant $(AB)^{-\lambda/2}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is proved to be best possible. Yang had yet the following result in paper[20]:

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+C)^{\lambda}}$$
$$<\frac{1}{(AB)^{\lambda/2}}B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{C}{2A}\right)^{1-\lambda}a_n^2\sum_{n=0}^{\infty}\left(n+\frac{C}{2B}\right)^{1-\lambda}b_n^2\right\}^{1/2}, \tag{15}$$

where $A,B,C>0$ and $0<\lambda\le 2$, and the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)(AB)^{-\lambda/2}$ is also proved to be best possible.

For the inequality (4), Yang and Debnath[21] gave a generalization as

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}<k_{\lambda}(p)\left\{\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{1/q},$$

where the constant $k_{\lambda}(p)=B\left(\frac{p-2+\lambda}{p},\frac{q-2+\lambda}{q}\right)$ (with $2-\min(p,q)<\lambda\le 2$) is proved to be best possible, and $B(m,n)$ is beta function.

Recently, Kuang and Debnath[22] gave general form on Hardy-Hilbert's inequality.

Let $a_n,\ b_n\ge 0, p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\sum_{n=0}^{\infty}(n+\lambda)^{1-t}a_n^p<+\infty, 0<\sum_{n=0}^{\infty}(n+\lambda)^{1-t}b_n^q<\infty, \frac{1}{2}\le\lambda\le\frac{1}{2}\min(p,q)$, and let $K(x,\ y)$ be a nonnegative and homogeneous function of degree $-t(t>0)$. If $K(1,\ y)$has its first four derivatives continuous on $(0,\ +\infty)$, and $(-1)^n K^{(n)}(1,\ y)\ge 0$ for $n=1,\ 2,\ 3,\ 4,\ K^{(m)}(1,\ y)y^{-2\lambda/r}\to 0,\ y\to+\infty$ for $m=0,\ 1,$

$$I(r,\ \lambda)=\int_0^{\infty}K(1,\ u)\,u^{-2\lambda/r}du<+\infty,\quad r=p,\ q,$$

then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,\ n+\lambda)\,a_m b_n<\left\{\sum_{m=0}^{\infty}[I(q,\ \lambda)-\varphi(q,\ m,\ t,\ \lambda)](m+\lambda)^{1-t}a_m^p\right\}^{1/p}\times$$
$$\left\{\sum_{m=0}^{\infty}\left[I(p,\ \lambda)-\varphi(p,\ m,\ t,\ \lambda)(m+\lambda)^{1-t}b_m^q\right]\right\}^{1/q},$$

where

$$\varphi(r,m,t,\lambda)=\left(\tfrac{\lambda}{m+\lambda}\right)^{1-\frac{2\lambda}{r}}\left\{K\left(1,\frac{\lambda}{m+\lambda}\right)\left[\frac{1}{1-2\lambda/r}-\frac{1}{2\lambda}\left(1+\frac{1}{3r}\right)\right]-\right.$$
$$\left.\frac{1}{24\lambda(m+\lambda)}K'\left(1,\frac{1}{m+\lambda}\right)\right\}>0$$

and $r=p,q$.

Newly, concerning the inequalities (3) and (4) , Yang and Debnath[23] established again new inequalities by introducing parameters A, B and λ:

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^{\lambda}} < \frac{k_{\lambda}(p)}{A^{\varphi_{\lambda}(P)} B^{\varphi_{\lambda}(q)}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right\}^{1/q},$$

where the constant factor $k_{\lambda}(p)/A^{\varphi_{\lambda}(p)} B^{\varphi_{\lambda}(q)}$ is best possible. In particular,

(i) for $\lambda = 1, A, B > 0$, it follows that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{Am+Bn} < \frac{\pi}{A^{1/q} B^{1/p} \sin \pi/p} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}; \tag{17}$$

(ii) for $\lambda = 2$, $A, B > 0$, it follows that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(Am+Bn)^2} < \frac{1}{AB} \left\{ \sum_{n=1}^{\infty} \frac{a_n^p}{n} \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \frac{b_n^q}{n} \right\}^{1/q}; \tag{18}$$

(iii) for $2 - min\{p, q\} < \lambda \le 2, A = B = 1$, it follows that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < k_{\lambda}(p) \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right\}^{1/q}, \tag{19}$$

where the constant factors in (17)–(19) are all the best possible.

For other form by introducing parameter λ, Yang[24] had the following results:

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < \frac{\pi}{\lambda \sin \pi/p} \left\{ \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q \right\}^{1/q},$$

where the constant factor $\pi/(\lambda \sin \pi/p)$ is best possible. In particular,

(i) For $\lambda = \frac{1}{2}$, it follows that

$$0 < \sum_{m,n=1}^{\infty} \frac{a_m b_n}{\sqrt{m}+\sqrt{n}} < \frac{2\pi}{\sin \pi/p} \left\{ \sum_{n=1}^{\infty} n^{(p-1)/2} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{(q-1)/2} b_n^q \right\}^{1/q};$$

(ii) For $p = q = \lambda = 2$, it follows that

$$0 < \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m^2+n^2} < \frac{\pi}{2} \left\{ \sum_{n=1}^{\infty} \frac{a_n^2}{n} \sum_{n=1}^{\infty} \frac{b_n^2}{n} \right\}^{1/2},$$

where the constant factors in the inequalities are best possible.

For the corresponding integral inequalities, some nice results with parameters were also obtained.

In 1998, Yang[25] introduced firstly parameters T and t to estimate the weight function such that the new inequality was built:

$$\int_0^T\int_0^T \frac{f(x)\,g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y \le B\left(\frac{t}{2},\ \frac{t}{2}\right)\left\{\int_0^T\left(1-\frac{1}{2}\left(\frac{x}{T}\right)^{t/2}\right)x^{1-t}f^2(x)\,\mathrm{d}x\right\}^{1/2}\times$$
$$\left\{\int_0^T\left(1-\frac{1}{2}\left(\frac{x}{T}\right)^{t/2}\right)x^{1-t}g^2(x)\,\mathrm{d}x\right\}^{1/2},$$

where $T,\ t>0$, and $B(p,\ q)$ is beta function.

At the same time, Yang got yet other results in the paper[26].

Let $b>a>0$, and $0<t\le 1, f,g\in L^2[0,\ +\infty)$. Then

$$\int_a^b\int_a^b \frac{f(x)\,g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y \le B\left(\frac{t}{2},\ \frac{t}{2}\right)\left(1-(a/b)^{1/4}\right)\left\{\int_a^b x^{1-t}f^2(x)\,\mathrm{d}x\int_a^b x^{1-t}g^2(x)\,\mathrm{d}x\right\}^{1/2}$$

and

$$\int_a^\infty\int_a^\infty \frac{f(x)g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y \le B\left(\frac{t}{2},\frac{t}{2}\right)\left\{\int_a^\infty\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{1}{2}}\right]x^{1-t}f^2(x)\mathrm{d}x\times\right.$$
$$\left.\int_a^\infty\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{1}{2}}\right]x^{1-t}g^2(x)\mathrm{d}x\right\}^{1/2},$$

where $B(m,n)$ is beta function.

In 1999, for integral form of Hardy-Hilbert's inequality Kuang[27] originated by introducing parameter t two new inequalities:

$$\int_a^b\int_a^b \frac{f(x)\,g(y)}{x^t+y^t}\mathrm{d}x\mathrm{d}y \le \left\{\omega(t,\ p,\ q)\int_a^b x^{1-t}f^p(x)\,\mathrm{d}x\right\}^{1/p}\times$$
$$\left\{\omega(t,\ q,\ p)\int_a^b x^{1-t}g^q(x)\,\mathrm{d}x\right\}^{1/q}, \tag{20}$$

where $\omega(t,\ p,\ q)=\frac{\pi}{t\sin\ \pi/pt}-\varphi(q)$ and here function φ is defined by

$$\varphi(r)=\int_0^{a/b}\frac{u^{t-2+1/r}}{1+u^t}\mathrm{d}u,\quad r=p,\ q.$$

And

$$\int_a^b\int_a^b \frac{f(x)\,g(y)}{(x+y)^t}\mathrm{d}x\mathrm{d}y \le \left\{B\left(\tfrac{1}{p},\ \tfrac{t-1}{p}\right)-\varphi_1(q)\right\}^{1/p}\left\{B\left(\tfrac{1}{q},\ \tfrac{t-1}{q}\right)-\varphi_1(p)\right\}^{1/q}\times$$
$$\left\{\int_a^b x^{1-t}f^p(x)\,\mathrm{d}x\right\}^{1/p}\left\{\int_a^b x^{1-t}g^q(x)\,\mathrm{d}x\right\}^{1/q}, \tag{21}$$

where $\max\left\{\frac{1}{p},\frac{1}{q}\right\}<t<b, 0<a<b, B(p,q)$ is beta function, and the function φ_1 is defined by

$$\varphi_1(r)=\int_0^{a/b}\frac{u^{-1/r}+u^{t-2+1/r}}{(1+u)^t}\mathrm{d}u,\quad r=p,q.$$

In the paper [28], Yang obtained the following result:

$$\int_{-\beta}^{\infty}\int_{-\beta}^{\infty}\frac{f(x)\,g(y)}{(x+y+2\beta)^{\lambda}}\mathrm{d}x\mathrm{d}y<k_{\lambda}(p)\left\{\int_{-\beta}^{\infty}(t+\beta)^{1-\lambda}f^{p}(t)\,\mathrm{d}t\right\}^{1/p}\left\{\int_{-\beta}^{\infty}(t+\beta)^{1-\lambda}g^{q}(t)\,\mathrm{d}t\right\}^{1/q}$$

and

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{(x+y+2\beta)^{\lambda}}<\left\{\int_{\alpha}^{\infty}\omega(\lambda,q,t)\,f^{p}(t)\,\mathrm{d}t\right\}^{1/p}\left\{\int_{\alpha}^{\infty}\omega(\lambda,p,t)\,g^{q}(t)\,\mathrm{d}t\right\}^{1/q}\quad(\alpha>-\beta),$$

where $\omega(\lambda,\ r,\ t)=k_{\lambda}(p)-\theta_{\lambda}(r)\left(\frac{\alpha+\beta}{t+\beta}\right)^{1+(\lambda-2)/r}(t+\beta)^{1-\lambda}$, $\theta_{\lambda}(r)=\int_0^1\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{(2-\lambda)/r}\mathrm{d}u$ and $k_{\lambda}(p)=B\left(\frac{p-2+\lambda}{p},\ \frac{q-2+\lambda}{q}\right)$ is best possible, and here $B(m,\ n)$ is beta function.

Lately, the new results were given by introducing parameters A, B and λ in the paper[23]:

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)\,g(y)}{(Ax+By)^{\lambda}}<\frac{k_{\lambda}(p)}{A^{\varphi_{\lambda}(p)}B^{\varphi_{\lambda}(q)}}\left\{\int_{0}^{\infty}x^{1-\lambda}f^{p}(x)\,\mathrm{d}x\right\}^{1/p}\left\{\int_{0}^{\infty}x^{1-\lambda}g^{q}(x)\,\mathrm{d}x\right\}^{1/q},\quad(22)$$

where the constant factor $k_{\lambda}(p)/A^{\varphi_{\lambda}(p)}B^{\varphi_{\lambda}(q)}$ is best possible, and the functions $\varphi_{\lambda}(r)$ and $k_{\lambda}$are defined respectively by

$$\varphi_{\lambda}(r)=(r+\lambda-2)/r\quad(r=p,\ q)\quad\text{and}\quad k_{\lambda}(p)=B(\varphi_{\lambda}(p),\ \varphi_{\lambda}(q)),$$

where $B(u,\ v)$ is the beta function.

In particular,

(i) for $\lambda=1$, A, $B>0$, it follows that

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)\,g(y)}{Ax+By}<\frac{\pi}{A^{1/q}B^{1/p}\sin\pi/p}\left\{\int_{0}^{\infty}f^{p}(x)\,\mathrm{d}x\right\}^{1/p}\left\{\int_{0}^{\infty}g^{q}(x)\,\mathrm{d}x\right\}^{1/q},\qquad(23)$$

(ii) for $\lambda=2$. A, $B>0$, it follows that

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)\,g(y)}{(Ax+By)^{2}}\mathrm{d}x\mathrm{d}y<\frac{1}{AB}\left\{\int_{0}^{\infty}x^{-1}f^{p}(x)\,dx\right\}^{1/p}\left\{\int_{0}^{\infty}x^{-1}g^{q}(x)\,\mathrm{d}x\right\}^{1/q},\qquad(24)$$

where the constant factors in (23) and (24) are still best possible.

Freshly, Kuang[29] established a new general form of Hilbert's inequality and its converses.

Let a_n, $b_n\geq 0$, α_n, $\beta_n>0$, $\frac{1}{p}+\frac{1}{q}=1$. For each positive integer $N<+\infty$ or $N=+\infty$, define

$$f_N(x)=e^{-x}\sum_{m=0}^{N}\frac{a_m x^{\alpha_m-1/2}}{\Gamma\left(\alpha_m+\frac{1}{2}\right)}\quad\text{and}\quad g_N(x)=e^{-x}\sum_{n=0}^{N}\frac{b_n x^{\beta_n-1/2}}{\Gamma\left(\beta_n+\frac{1}{2}\right)}.$$

If $1<p<+\infty$, then

$$\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{a_m b_n}{\alpha_m+\beta_n}\leq\frac{\pi}{\sin\pi/p}\left\|f_N\right\|_p\ \left\|g_N\right\|_q,$$

where the interval of the integral is $(0,+\infty)$.

If $0 < p < 1$, then the inequality is reversed.

Basing on the above-mentioned results, some important corollaries were established:

Corollary 1 *With the assumptions as the above described, then*

$$\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{a_m b_n}{m+n+1} \le \frac{\pi}{\sin \pi/p}\|f_N\|_p\|g_N\|_q \le \frac{\pi}{\sin \pi/p}\|a\|_p\|b\|_q.$$

Corollary 2 *With the assumptions as the above described, Define the function by*

$$f_{N,\lambda}(x)=\begin{cases} e^{-x}\sum\limits_{m=0}^{N}\dfrac{a_m x^{\alpha_m^\lambda-1/2}}{\Gamma\left(\alpha_m^\lambda+\frac{1}{2}\right)} & (\lambda \ge 1) \\ e^{-x}\sum\limits_{m=0}^{N}\dfrac{a_m x^{2^{\lambda-1}\alpha_m^\lambda-\frac{1}{2}}}{\Gamma\left(2^{\lambda-1}\alpha_m^\lambda+\frac{1}{2}\right)} & (0<\lambda<1)\end{cases}$$

$g_{N,\lambda}(x)$ may be similarly defined. If $1 < p < +\infty$, then

$$\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{a_m b_n}{(\alpha_m+\beta_n)^\lambda} \le \frac{\pi}{\sin \pi/p}\|f_{N,\lambda}\|_p\|g_{N,\lambda}\|_q.$$

If $0 < p < 1$, then the inequality is reversed.

Corollary 3 *Let $b_{m_k} \ge 0, \alpha_{m_k} > 0, 1 < p_k < +\infty, \sum_{k=1}^{n} 1/p_k = 1$ and $\lambda \ge 1$. Define*

$$f_{N,k}(x) = e^{-x}\sum_{m_k=0}^{N}\frac{b_{m_k}x^{\alpha_{m_k}^\lambda-1/2}}{\Gamma\left(\alpha_{m_k}^\lambda+\frac{1}{2}\right)},$$

then

$$\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\cdots\sum_{m_n=0}^{N}\frac{b_{m_1}b_{m_2}\cdots b_{m_n}}{(\alpha_{m_1}+\alpha_{m_2}+\cdots+\alpha_{m_n})^\lambda} \le \prod_{k=1}^{n}\Gamma(1-1/p_k)\|f_{N,k}\|_{p_k}.$$

If $0<\lambda<1$ and $\alpha_{m_k}^\lambda (m_k=0,1,2,\cdots,N)$ are replaced by $n^{\lambda-1}\alpha_{m_k}^\lambda$, then the inequality is reversed.

In particular, when $\alpha_{m_k} = m_k + \frac{1}{2}$, the following inequality is valid:

$$\begin{aligned}\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\cdots\sum_{m_k=0}^{N}\frac{b_{m_1}b_{m_2}\cdots b_{m_n}}{\left(m_1+m_2+\cdots+m_n+\frac{n}{2}\right)^\lambda} &\le \prod_{k=1}^{n}\Gamma(1-1/p_k)\|f_{N,k}\|_{p_k} \\ &\le \prod_{k=1}^{N}\Gamma(1-1/p_k)\|b_{m_k}\|_{p_k}.\end{aligned}$$

Recently, the Hardy-Hilbert inequality was generalized to multiple integral with parameter t by Hong[30].

Let $a>0, b\ge 0, 1/p_1+1/p_2+\cdots+1/p_n=1, p_i>1, f_i\ge 0, r_i>b$ and $t>(1/a)(n-1-b/r_i)$ $(r_i=$

$p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_n)$, $i=1,2,\cdots,n$. Then

$$\int_\alpha^\infty\int_\alpha^\infty\cdots\int_\alpha^\infty \frac{1}{\left(\sum\limits_{i=1}^n (x_i-\alpha)^a\right)^t}\prod_{i=1}^n f_i(x_i)\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n \le \frac{\Gamma^{n-2}(1/a)}{a^{n-1}\Gamma(t)}\times$$

$$\prod_{i=1}^n\left\{\Gamma\left(\frac{1}{a}(1-b/r_i)\right)\Gamma\left(t-\frac{1}{a}(n-1-b/r_i)\right)\int_\alpha^\infty (x-\alpha)^{n-1-at} f_i^{p_i}(x)\,\mathrm{d}x\right\}^{1/p_i},$$

where $\Gamma(x)$is gamma function.

Other extension of Hardy-Hilbert's integral inequality was established by He, Yu and Gao[31]. We enumerate it as follows:

Let $x^{1-(\lambda+1)/n}f_i(x)\in L^n(0,+\infty), i=1, 2, \cdots, n$. If $(n-1)(1-1/s)<\lambda<n, s>1$ and $n\ge 2$, then

$$\int\cdots\int_{R_+^n}\frac{f_1(x_1)f_2(x_2)\cdots f_n(x_n)}{(x_1+x_2+\cdots+x_n)^\lambda}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n \le \frac{\Gamma^n(\lambda/n)}{\Gamma(\lambda)}\prod_{i=1}^n\left\{\int_0^\infty x^{n-\lambda-1}f_i^n(x)\,\mathrm{d}x\right\}^{1/n},$$

where $R_+=(0,+\infty)$, $\Gamma(x)$ is gamma function, and the coefficient $\frac{\Gamma^n(\lambda/n)}{\Gamma(\lambda)}$ is best possible.

Very recently, Kuang[29] gave the general form on Hardy-Hilbert's integral inequality:

Let f and g be nonnegative measurable functions on $(0, a)$, $\alpha(x)$ and $\beta(y)$ be positive measurable on $(0, a)$, $\frac{1}{p}+\frac{1}{q}=1$. For each positive number $a<+\infty$ or $a=+\infty$, define two functions by

$$F(u)=e^{-u}\int_0^a\frac{u^{\alpha(x)-1/2}}{\Gamma\left(\alpha(x)+\frac{1}{2}\right)}f(x)\,\mathrm{d}x \text{ and } G(u)=e^{-u}\int_0^a\frac{u^{\beta(y)-1/2}}{\Gamma\left(\beta(y)+\frac{1}{2}\right)}g(y)\,\mathrm{d}y.$$

If $1<p<+\infty$, then $\int_0^a\int_0^a\frac{f(x)g(y)}{\alpha(x)+\beta(y)}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{\sin\pi/p}\|F\|_p\|G\|_q$;

If $0<p<1$, then the inequality is reversed.

Basing on the above mentioned results, some important corollaries were established by Kuang[29].

Corollary 1 *With the assumptions as the above described, if* $1<p<+\infty$ *then*

$$\int_0^a\int_0^a\frac{f(x)g(y)}{(\alpha(x)+\beta(y))^\lambda}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{\sin\pi/p}\|F\|_p\|G\|_q,$$

where the functions F *and* G *are defined by*

$$F(u)=\begin{cases} e^{-u}\int_0^a\frac{u^{[\alpha(x)]^\lambda-\frac{1}{2}}}{\Gamma\left([\alpha(x)]^\lambda+\frac{1}{2}\right)}f(x)\,\mathrm{d}x & (\lambda\ge 1)\\ e^{-u}\int_0^a\frac{u^{2^{\lambda-1}}[\alpha(x)]^\lambda-\frac{1}{2}}{\Gamma\left(2^{\lambda-1}[\alpha(x)]^\lambda+\frac{1}{2}\right)}f(x)\,\mathrm{d}x & (0<\lambda<1).\end{cases}$$

The substitute of $\beta(y)$, $g(y)$ for $\alpha(x)$, $f(x)$ in the above expression, gives$G(u)$.

If $0 < p < 1$, then the inequality is reversed.

Corollary 2 *Let* $\{f_k\}$ *be nonnegative measurable functions on* $[0, a]$, *and* $\{\alpha_k\}$ *be positive and measurable on* $[0, a]$, $1 < p_k < +\infty$, $\sum_{k=1}^n \frac{1}{p_k} = 1$. *Define a function* F *by*

$$F_k(u) = e^{-u} \int_0^a \frac{u^{(\alpha_k(x))^\lambda - 1/2}}{\Gamma\left((\alpha_k(x))^\lambda + \frac{1}{2}\right)} f_k(x)\,\mathrm{d}x.$$

If $\lambda \geq 1$, then

$$\int_0^a \int_0^a \cdots \int_0^a \frac{f_1(x_1) f_2(x_2) \cdots f_n(x_n)}{(\alpha_1(x_1) + \alpha_2(x_2) + \cdots + \alpha_n(x_n))^\lambda} \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \leq \prod_{k=1}^n \Gamma(1 - 1/p_k) \|F_k\|_{p_k}.$$

If $0 < \lambda < 1$, and $\{\alpha_k(x)\}^\lambda$ are replaced by $n^{\lambda-1}\{\alpha_k(x)\}^\lambda$, then the inequality is reversed.

4. New inequalities being applied

The primary proof on Hilbert's inequality is based on the following identity (see [1] §9.6, formula (9.6.2)):

$$\int_{-\pi}^{\pi} t \left\{ \sum_{r=1}^n (-1)^r (a_r \cos rt - b_r \sin rt) \right\}^2 \mathrm{d}t = 2\pi (S - T),$$

where $S = \sum_{r=1}^n \sum_{s=1}^n \frac{a_r b_s}{r+s}$ and $T = \sum_{r=1}^n \sum_{s=1}^n \frac{a_r b_s}{r-s}$ $(r \neq s)$.

In 1992, Hu Ke[32] built an elegant inequality:

$$|S|^2 + |T|^2 \leq \pi^2 \sum_{r=1}^n |a_r|^2 \sum_{s=1}^n |b_s|^2. \tag{25}$$

The inequality (25) is obviously a new refinement on Hilbert's double series theorem.

Hu applied the fundamental inequality (see below)established by himself to attain some nice results. We state them as follows:

Let's denote that $S_{i,\lambda}(a,b) = \sum_{r,s=0}^n a_r b_s \big/ (r+s+\lambda)^i$, $T_{i,\lambda}(a,b) = \sum_{r\neq s, r,s=0}^n a_r b_s \big/ (r-s-\lambda)^i$. $\|x\|^2 = \sum_{i=0}^n |x_i|^2$. Hu[33] proved that

$$|S_{1,1}(a,\ b)|^2 + |T_{1,0}(a,\ b)|^2 \leq \pi^2 \left\{ \|a\|^2 \|b\|^2 (1 - A^2)^{1/2} \right\},$$

where A is a real number.

In 1996, Actually, Hu got the general result with the parameter λ[35]. In particular, when $\lambda = \frac{1}{2}$, Hu[34] got that

$$|S_{1,\,1/2}(a,\ b)|^2 + |T_{1,0}(a,\ b)|^2 \leq \pi^2 \left\{ \left[\|a\|^2 \|b\|^2 - (1/\pi^4) S_{2,\,1/2}(a,\ \bar{a}) S_{2,\,1/2}(b,\ \bar{b}) \right]^2 - (1/\pi^8) \left[\|\tilde{a}\|^2 S_{2,\,1/2}(b,\ \bar{b}) + \left\|\tilde{b}\right\|^2 S_{2,\,1/2}(b,\ \bar{b}) \right]^2 \right\}^{1/2},$$

where $\|\tilde{x}\|^2 = \sum_{k=0}^{n} |x_k|^2 \Big/ (k+1)$, and when $\lambda = 1$, Hu[35] got that

$$\left\{|T_{1,0}(a,\ b)|^2 + |S_{1,1}(a,b)|^2\right\}^2 + \pi^2 \left\{S_{2,1}(a,\bar{a})\,\|b\|^{1/2} + S_{2,1}(b,\bar{b})\,\|a\|^{1/2}\right\}^2 \le \pi^4 (\|a\|\ \|b\|)^4 .$$

Let $\lambda \neq 0$ and λ is not a negative integer. Hu[35] gave the following result:

$$\begin{aligned} &|T_{1,0}(a,\ b)|^2 + (\sin^2\lambda\pi)\,|(\cot\lambda\pi)\,S_{1,\lambda}(a,\ b) - (1/\pi)\,S_{2,\lambda}(a,\ b)|^2 \\ &\le \pi^2 \left\{\left[\|a\|^2\,\|b\|^2 - (\pi\sin\lambda\pi)^2\,S_{1,\lambda}(a,\ \bar{a})\,S_{1,\lambda}(b,\ \bar{b})\right]^2 - \right. \\ &\left. \left[(4/\pi^4)\,(\omega_\lambda(a,\ b) + \omega_\lambda(b,\ a))\right]^2\right\}^{1/2}, \end{aligned}$$

where

$$\begin{aligned} \omega_\lambda(a,\ b) = (\sin\lambda\pi)\left\{\|b\|^2\left[(\pi/3)\,S_{1,\lambda}(a,\bar{a}) + (\cot\lambda\pi)\,S_{2,\lambda}(a,\bar{a}) - \pi^{-2}S_{3,\lambda}(a,\bar{a})\right] - \right. \\ \left. \pi^{-1}S_{1,\lambda}(b,\bar{b})\,T_{2,0}(a,\bar{a})\right\}. \end{aligned}$$

It is simultaneously refinement on Hilbert's inequality and Ingham's inequality.

When λ is a positive integer, Hu[35] gave that

$$|T_{1,0}(a,b)|^2 + |S_{1,\lambda}(a,b)|^2 \le \pi^2 \left\{\left[\|a\|^2\,\|b\|^2\right] - \left(\frac{4}{\pi}\right)^2 \left[S_{2,\lambda}(b,\bar{b})\,\|a\|^2 + S_{2,\lambda}(a,\bar{a})\,\|b\|^2\right]^2\right\}^{\frac{1}{2}} .$$

When $\lambda \neq 0, \pm 1, \pm 2, \cdots$, Hu[36] proved recently that

$$|T_{1,\lambda}(a,\ b)|^2 + |S_{1,-\lambda}(a,\ b)|^2 \le \frac{\pi^2}{\sin^2\lambda\pi}\,\|a\|^2\,\|b\|^2 - \|b\|^2 \left|\sum_{k=1}^{n} a_k/(k-\lambda)\right|^2 .$$

This is a refinement on the Polya-Szego inequality of the form

$$\left.\begin{aligned} |S_{1,-\lambda}(a,\ b)|^2 \\ |T_{1,\lambda}(a,\ b)|^2 \end{aligned}\right\} \le \frac{\pi^2}{\sin^2\lambda\pi}\,\|a\|^2\,\|b\|^2 .$$

In 1999, Gao[37] originated a new inequality by making use of the positive definiteness of Gram matrix:

$$(\alpha,\beta)^2 \le \|\alpha\|^2\,\|\beta\|^2 - (\|\alpha\|\,x - \|\beta\|\,y)^2, \tag{26}$$

where $x = (\beta,\gamma), y = (\alpha,\gamma)$ with $\|\gamma\| = 1$ and $xy \ge 0$ (by the way, this condition should be added in the paper[35]). Using the inequality (26), Gao[38] built again a new inequality which is stronger than the inequality (25):

$$\left|\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}\right|^2 + \left|\sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{a_m b_n}{m-n}\right|^2 \le \pi^2(1-r)\sum_{n=1}^{\infty}|a_n|^2\sum_{n=1}^{\infty}|b_n|^2,$$

where $r = \frac{1}{\pi^2}\left(s^2(a)/\|a\|^2 + s^2(b)/\|b\|^2 \right), s(x) = \sum_{n=1}^{\infty} x_n/n$ and $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2$.

Before long, Gao (see [39], p.197-204) applied the inequality (26) to prove the following inequality:

$$\left|\sum_{m,n=0}^{\infty} \frac{a_m b_n}{m+n+1}\right|^2 + \left|\sum_{\substack{m,n=0 \\ m\neq n}}^{\infty} \frac{a_m b_n}{m-n}\right|^2 < \pi^2(1-c)\sum_{n=0}^{\infty}|a_n|^2\sum_{n=0}^{\infty}|b_n|^2, \tag{28}$$

where $c = (1/2\pi^2)\left\{|s(a)|^2/\|a\|^2 + |s(b)|^2/\|b\|^2\right\}$, $\|x\| = \left(\sum_{n=0}^{\infty}|x_n|^2\right)^{1/2}$ and the function $s(x)$ is defined by

$$s(x) = x_0 + \sum_{n=1}^{\infty}\frac{(2n+1)x_n}{n(n+1)}.$$

Similarly, in the paper [38] an improvement of the original Hilbert integral inequality was established by means of (26):

$$\left\{\iint_{\mathrm{R}_+^2}\frac{f(x)g(y)}{x+y}dxdy\right\}^2 < \pi^2(1-R)\|f\|^2\|g\|^2,$$

where $f,\ g \in L^2(\mathrm{R}_+), \mathrm{R}_+ = (0,\ +\infty)$, $R = (1/\pi)\left(x/\|g\| - y/\|f\|\right)^2$ with $x = (2/\pi)^{1/2}(g,\ e)$ and $y = (2\pi)^{1/2}(f,\ e^{-s})$, e being the exponential integral with parameter t:

$$e(t) = \int_{\mathrm{R}_+}\frac{e^{-s}}{s+t}\mathrm{d}s \quad (\ t\ \in\ \mathrm{R}_+).$$

At the same time, the general result was given by the paper [40]:

$$\left\{\int_0^{\infty}\int_0^{\infty}\frac{f(s)g(t)}{s+t}\right\}^2 \le \pi^2\int_0^{\infty}f^2(t)\,\mathrm{d}t\int_0^{\infty}g^2(t)\,\mathrm{d}t - G(\xi,\eta,\zeta) \tag{29}$$

where $G(\xi,\ \eta,\ \zeta) \ge 0$. And the equality in (29) holds if and only if ξ, η and ζ are linearly dependent.

We recommend specially Hu Ke's work here.

Twenty years ago, Hu[41] built an important inequality of the form

$$\sum_n A_nB_n \le \left(\sum_n B_n^q\right)^{\frac{1}{q}-\frac{1}{p}}\left\{\left(\sum_n B_n^q\sum_n A_n^p\right)^2 - \left(\sum_n B_n^q e_n\sum_n A_n^p - \sum_n B_n^q\sum_n A_n^p e_n\right)^2\right\}^{\frac{1}{2p}}, \tag{30}$$

where $p \ge q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, A_n, $B_n \ge 0$ and $1 - e_n + e_m \ge 0$.

The corresponding integral form of (30) is that

$$\int F(x)G(x)\,dx \le \left\{\int G^q(x)\,\mathrm{d}x\right\}^{1/q-1/p}\left\{\left(\int G^q(x)\mathrm{d}x\int F^p(x)\mathrm{d}x\right)^2 - \left(\int G^q(x)e(x)\mathrm{d}x\int F^p(x)\mathrm{d}x - \int G^q(x)\mathrm{d}x\int F^p(x)e(x)\mathrm{d}x\right)^2\right\}^{\frac{1}{2p}}. \tag{31}$$

Evidently, by using the inequalities (30) and (31), Hardy-Hilbert's inequalities may be improved[32,33]. To be specific, in 1979, Hu[42] gave the following result:

$$\left|\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\right|^2$$
$$\leq \pi^2\left\{\left(\int_0^\infty f^2(x)\,\mathrm{d}x\right)^2\left[1-\left(\int_0^\infty E(x)f^2(x)\,\mathrm{d}x\right)^2\left(\int_0^\infty f^2(x)\,\mathrm{d}x\right)^{-2}\right]\right\}^{1/2}\times$$
$$\left\{\left(\int_0^\infty g^2(y)\,\mathrm{d}y\right)^2\left[1-\left(\int_0^\infty E(y)g^2(y)\,\mathrm{d}y\right)^2\left(\int_0^\infty g^2(y)\,\mathrm{d}y\right)^{-2}\right]\right\}^{1/2}.$$

where $E(x) = \frac{2}{\pi}\int_0^\infty \frac{e(xt^2)}{1+t^2}\mathrm{d}t - e(x), (1-e(x)+e(y)\geq 0)$. In particular, when $e(x) = \frac{1}{2}\cos\sqrt{x}$, $E(x) = \frac{1}{2}\left(e^{-\sqrt{x}}-\cos\sqrt{x}\right)$.

Afterward Hu[32] gave again its applications and the proof of it was given in detail.

In 1994, Hu improved and extended the Hardy-Litlewood-Polya inequality[43]:

Let $p,q>1, 0<\lambda\leq 1, \lambda = 2-\frac{1}{p}-\frac{1}{q}, (K(x,\ y))^{1/\lambda}$ is a homogeneous express with degree -1. If $f\in L^p(0,\infty), g\in L^q(0,\infty)$, $1-e(x)+e(y)\geq 0$ and denote that $\frac{1}{q'} = 1-\frac{1}{q}$ and $\int_0^\infty \left\{K(x,\ 1)\,x^{-1/q'}\right\}^{1/\lambda}\mathrm{d}x = k$, then

$$\int_0^\infty\int_0^\infty K(x,y)\,f(x)\,g(y)\,\mathrm{d}x\mathrm{d}y$$
$$\leq k^\lambda\left\{\int_0^\infty f^p(x)\,\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)\,\mathrm{d}y\right\}^{\frac{1}{q}}\left\{\begin{array}{ll}\left(1-R_1^2\right)^{1/2q'} & q'>q\\ \left(1-R_2^2\right)^{1/2q} & q>q'\end{array}\right.$$

where

$$R_1 = \left(\int_0^\infty f^p(x)\,E_1(x)\,\mathrm{d}x\right)\left(\int_0^\infty f^p(x)\,\mathrm{d}x\right)^{-1} - \left(\int_0^\infty g^q(y)\,e(y)\,\mathrm{d}y\right)\left(\int_0^\infty g^q(y)\,\mathrm{d}y\right)^{-1},$$

$$R_2 = \left(\int_0^\infty f^p(x)\,e(x)\,\mathrm{d}x\right)\left(\int_0^\infty f^p(x)\,\mathrm{d}x\right)^{-1} - \left(\int_0^\infty g^q(y)\,E_1(y)\,\mathrm{d}y\right)\left(\int_0^\infty g^q(y)\,\mathrm{d}y\right)^{-1},$$

$$kE_1(x) = \int_0^\infty e(x/\omega)\left\{K(\omega,1)\,\omega^{-1/q'}\right\}^{1/\lambda}\mathrm{d}\omega.$$

When $\lambda = 1, p = q$, an improvement of Hilbert's inequality is obtained.

Recently, Hu built again a new inequality[36]. In order to make it convenient for applications, we introduce the result of the paper[36] as follows:

Let $x = (x_1, x_2, \cdots, x_n)$, a_k, b_k, $c_k^{(i)} \in C\,(k = 1,2,\cdots i = 1,2,\cdots), e_k\in R, r, s>0$.

$$(a^r, b^s) = \sum_{k=1}^n a_k^r\bar{b}_k^s,\ \|a\|_p = \left(\sum_{k=1}^n |a_k|^p\right)^{1/p},\ \|a\|_2 = \|a\|,\ (|x|^p, e) = \sum_{k=1}^n |x_k|^p e_k,$$

$$S_p(x,c) = \frac{\left|\left(x^{p/2}, c\right)\right|}{\|x\|_p^{p/2}},\ T_p(a,b,c) = S_p(a,c) - S_q(b,c),\ R_p(a,b,e) = \frac{(|a|^p, e)}{\|a\|_p^p} - \frac{(|b|^q, e)}{\|b\|_q^q}.$$

If $\frac{1}{p}+\frac{1}{q}=1, p>1, 1-e_n+e_m\geq 0$, $\left\|c^{(i)}\right\|=1$ and $m\left(\frac{1}{p}\right)=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$,then

$$|(a,b)|\leq\|a\|_p\|b\|_q\left(1-\omega_{m,p}^{(2)}(a,b,c,e)\right)^{m(1/p)}, \tag{32}$$

where

$$\omega_{m,p}^{(2)}(a,b,c,e)=T_p^2\left(a,b,c^{(m)}\right)+\sum_{i=1}^{m-1}T_p^2\left(a,b,c^{(i)}\right)\prod_{k=i+1}^{m}S_p\left(a,c^{(k)}\right)S_q\left(b,c^{(k)}\right)+$$
$$\frac{1}{2}R_p^2(a,b,e)\prod_{k=1}^{m}S_p\left(a,c^{(k)}\right)S_q\left(b,c^{(k)}\right).$$

When $m=1$, the term with Σ disappears, and $a_k, b_k\geq 0$ for case $p\neq 2$; $a_k, b_k, c_k\in C$ for case $p=2$. In particular, if $S_p(a,c)S_q(b,c)<1$, then

$$|(a,b)|\leq\|a\|_p\|b\|_q\left\{1-\frac{1}{S_p(a,\ c)S_q(b,c)}\left(S_p(a,c)-S_q(b,c)\right)^2\right\}^{m\left(\frac{1}{p}\right)}. \tag{33}$$

In particular, when $p=2$, the inequalities (30)–(33) are improvements on Hölder's inequality (Cauchy's inequality included). Obviously, applying these results to estimate respectively the right-hand sides of (1)–(6) might obtain some new results. Actually, these works have been being done. Some new results have just appeared in the latest papers[44,45]. And some new refinements and extensions or generalizations on Hardy-Hilbert's inequalities will appear in times to come.

Finally, we point out that the treatment of Hilbert's inequality by means of geometric method is also an important way (cf. [46] etc.).

5. Some open problems/questions

Here we enumerate several open questions or unsolved problems for reference.

1. Let $S=\sum_{l,m=1}^{n}a_lb_m/(l+m)$ or $S=\sum_{l,m=0}^{n}a_lb_m/(l+m+1), T=\sum_{l\neq m}^{n}a_lb_m/(l-m)$. If S and T were made to estimate respectively, then there are the best value θ[13–15]. If we try to estimate $\left(|S|^2+|T|^2\right)$, then what is the best value θ? (Proposed by Hu Ke).

2. Could there be found some interesting applications of Hilbert-type inequalities (generalizations and refinements) in mathematical science? (Proposed by L. C. Hsu).

3. How to construct some novel inequalities in the converse direction for Hilbert-type sums and integrals (other than those given by [29], etc.)? (Proposed by L. C. Hsu).

4. Let $U=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y, V=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x-y}\mathrm{d}x\mathrm{d}y\,(x\neq y), \|f\|^2=\int_0^\infty f^2(x)\,\mathrm{d}x$. If $0<\|f\|^2<+\infty$ and $0<\|g\|^2<+\infty$, then is the following inequality valid?

$$|U|^2+|V|^2\leq\pi^2\|f\|^2\|g\|^2.$$

(Proposed by Gao Mingzhe).

5. Let $\tilde{S}=\sum_{m,n=1}^{\infty}a_mb_n/(m+n)$ or $\tilde{S}=\sum_{m,n=0}^{\infty}a_mb_n/(m+n+1), \tilde{T}=\sum_{m\neq n}^{\infty}a_mb_n/(m-n)$, $\|x\|_k^2=\sum_{n=k}^{\infty}|x_n|^2, k=0,\ 1$. Show that if $0<\|a\|_k^2<+\infty$ and $0<\|b\|_k^2<+\infty$, then

$$|U|^2+|V|^2\leq\pi^2\|a\|_k^2\|b\|_k^2.$$

Demand: Don't employ the method in paper [38] (i.e. trigonometry). (Proposed by Gao Mingzhe).

6. K. Oleszkiewicz[46] proved the inequalities (1) and (3) with the help of the geometrical method. Could the inequalities (5) and (6) be proved by the geometrical method? (Proposed by Gao Mingzhe).

7. Find some applications of Hilbert's inequality in science and technology. (Proposed by Gao Mingzhe).

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Hilbert 不等式的各种精化与拓广综述

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摘要: 在本文中，我们较全面的概述了 1990 年至 2002 年间有关 Hilbert 级数型与积分型不等式的种种精化与拓广工作. 涉及发表于海内外的文献资料 40 余篇，其中我国学者的工作成果占有重要份量. 文末还列举了一些有待进一步研究的公开问题.

关键词: Hilbert 不等式; Hardy-Hilbert 不等式; 权函数; Gram 矩阵; 重级数.