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On Characterization of Iterative Approximation for Asymptotically Pseudocontractive Mappings

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Abstract: Let C be a nonempty bounded closed convex subset of a Banach space X, and $T: C \to C$ be uniformly L-Lipschitzian with $L \geq 1$ and asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. Fix $u \in C$. For each $n \geq 1$, x_n is a unique fixed point of the contraction $S_n(x) = (1 - \frac{t_n}{Lk_n})u + \frac{t_n}{Lk_n}T^n x \ \forall x \in C$, where $\{t_n\} \subset [0, 1)$. Under suitable conditions, the strong convergence of the sequence$\{x_n\}$to a fixed point of T is characterized.

Key words: fixed point; asymptotically pseudocontractive mapping; uniform Lipschitzian mapping; uniform normal structure; Banach contraction principle.
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1. Introduction

Let X be a real Banach space, X^* be the topological dual space of X, and $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*. Let C be a nonempty subset of X and $T: C \to C$ be a mapping of C into itself. Let $F(T)$ denote the set of all fixed points of T, and $J: X \to 2^{X^*}$ denote the normalized duality mapping defined by $J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x^*\| = \|x\|\}, x \in X$.

Definition 1.1 *A mapping $T: C \to C$ is said to be*

(1) asymptotically nonexpansive if there is a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n = 0, 1, 2, \cdots$;

(2) asymptotically pseudocontractive if there is a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\langle T^n x - T^n y, j(x-y) \rangle \leq k_n \|x - y\|^2$ for all $x, y \in C$, $j(x-y) \in J(x-y)$, and $n = 0, 1, 2, \cdots$.

In 1972, Goebel and Kirk[5] introduced the concept of asymptotically nonexpansive mapping. An early fundamental result due to Goebel and Kirk[5] showed that if X is a uniformly convex Banach space, C is a nonempty bounded closed convex subset of X, and $T: C \to C$ is an asymptotically nonexpansive mapping, then T has a fixed point in C. In 1991, Schu[9] introduced the concept of asymptotically pseudocontractive mapping. It is well known that the class of asymptotically pseudocontractive mappings is wider than the class of asymptotically

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nonexpan- sive mappings[5]. The existence and approximation problems for fixed points of nonexpansive mappings,asymptotically nonexpansive mappings and asymptotically pseudocontractive mappings were investigated extensively by Kirk[1], Goebel and Kirk[5], Schu[9], Xu and Tan[10], Lim and Xu[11], Chang[13,14] and the author[6,7].

Let C be a nonempty bounded closed convex subset of a real Banach space X, and $T : C \to C$ be an asymptotically nonexpansive mapping (we may always assume $k_n \geq 1$ for all $n \geq 1$). Fix a u in C and define for each integer $n \geq 0$ the contraction $S_n : C \to C$ by

$$S_n(x) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x, \tag{I}$$

where $\{t_n\} \subset [0, 1)$. Then the Banach Contraction Principle yields a unique point x_n fixed by S_n. In 1994, Lim and Xu[11] proved the following theorem concerning the iterative approximation problem for fixed points of asymptotically nonexpansive mappings.

Theorem 1.1[11, Theorem 2] *Suppose X is a uniform smooth Banach space and $\{t_n\}$ is chosen so that $\lim_{n\to\infty}(k_n - 1)/(k_n - t_n) = 0$. Suppose in addition the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ holds, where the sequence $\{x_n\}$ is defined by (I). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Remark 1.1 Theorem 1.1 is a partial generalization of Reich's convergence theorem[2] for approximating fixed points of nonexpansive mappings by the following iterative scheme

$$x_n = \frac{1}{n}u + (1 - \frac{1}{n})Tx_n,$$

where $\{x_n\}$ is said to be an approximating fixed point of T, i.e., it possesses the property that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

On the other hand, recently, Chang[14] established the strong convergence result on approximating fixed points of uniformly L-Lipschitzian and asymptotically pseudocontractive mappings.

The purpose of this paper is to extend Theorem 1.1 to the case of asymptotically pseudocontractive mappings. Let C be a nonempty bounded closed convex subset of a Banach space X, and $T : C \to C$ be uniformly L-Lipschitzian with $L \geq 1$ and asymptotically pseudo-contractive with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. Fix $u \in C$. For each $n \geq 1$, x_n is a unique fixed point of the contraction $S_n(x) = (1 - \frac{t_n}{Lk_n})u + \frac{t_n}{Lk_n}T^n x$ $\forall x \in C$, where $\{t_n\} \subset [0, 1)$. In other words, the sequence$\{x_n\}$is generated by the following iterative scheme:

$$x_n = (1 - \frac{t_n}{Lk_n})u + \frac{t_n}{Lk_n}T^n x_n, \quad \forall n \geq 1.$$

Under suitable conditions, the strong convergence of $\{x_n\}$ to a fixed point of T is characterized. It is remarkable that if T is nonexpansive, then by taking $L = 1$, $k_n = 1$ and $t_n = 1 - \frac{1}{n}$ for all $n \geq 1$, we have the iterative scheme $x_n = \frac{1}{n}u + (1 - \frac{1}{n})T^n x_n$ due to Reich[2].

2. Preliminaries

Let X be a real Banach space. Recall that X is said to be smooth if, for each $x \in S_X$, the unit sphere of X, the limit $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $y \in S_X$. In this case, the norm of X is said to be Gateaux differentiable. It is said to be uniformly Gateaux differentiable if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$. The norm is said to be Frechet differentiable if for each $x \in S_X$, this limit is attained uniformly for $y \in S_X$. Finally, the norm is said to be uniformly Frechet differentiable if the limit is attained uniformly for $(x, y) \in S_X \times S_X$. In this case X is said to be uniformly smooth. Since the dual X^* of X is uniformly convex if and only if the norm of X is uniformly Frechet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gateaux differentiable norm. The reverse is false.

Recall also that the normalized duality mapping$J : X \to 2^{X^*}$ is single-valued if and only if X is smooth. If X is smooth, then $J : X \to X^*$ is weakly sequentially continuous at zero, i.e., $\{J(x_n)\}$ converges to zero in the weak* topology $\sigma(X^*, X)$ of X^* as $\{x_n\}$ converges weakly to zero in X.

Let E be a nonempty bounded closed convex subset of a Banach space X, and let $d(E) = \sup\{\|x-y\| : x, y \in E\}$ be the diameter of E. For each $x \in E$, let $r(x, E) = \sup\{\|x-y\| : y \in E\}$ and let $r(E) = \inf\{r(x, E) : x \in E\}$, the Chebyshev radius of E relative to itself. The normal structure coefficient[8] of X is defined as the number

$$N(X) = \inf\{d(E)/r(E) : E \text{ bounded closed convex subset of } X \text{ with } d(E) > 0\}.$$

A space X such that $N(X) > 1$ is said to have uniformly normal structure. It is known[3] that a space with uniformly normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniformly normal structure.

Recall that a Banach limit LIM is a bounded linear functional on l^∞ such that

$$\|\mathrm{LIM}\| = 1, \quad \liminf_{n\to\infty} t_n \le \mathrm{LIM}_n t_n \le \limsup_{n\to\infty} t_n,$$

and $\mathrm{LIM}_n t_n = \mathrm{LIM}_n t_{n+1}$ for all $\{t_n\}$ in l^∞. Then we can define the real-valued continuous convex function φ on X by $\varphi(z) = \mathrm{LIM}_n\|x_n - z\|^2$ for each $z \in X$.

The following lemmas shall play important roles in the proofs of our main results.

Lemma 2.1[12,Lemma 1.2] *Let X be a Banach space with a uniformly Gateaux differentiable norm, C be a nonempty closed convex subset of X, and $\{x_n\}$ be a bounded sequence in X. Let* LIM *be a Banach limit and $y \in C$. Then $\mathrm{LIM}_n\|x_n - y\|^2 = \lim_{z\in C}\mathrm{LIM}_n\|x_n - z\|^2$ if and only if $\mathrm{LIM}_n\langle x - y, J(x_n - y)\rangle \le 0$ for all $x \in C$.*

Lemma 2.2 *Let X be a uniformly convex Banach space with a uniformly Gateaux differentiable norm, C a nonempty closed convex subset of X, and $\{x_n\}$ a bounded sequence of X. Then the set $D = \{u \in C : \mathrm{LIM}_n\|x_n - u\|^2 = \min_{z\in C}\mathrm{LIM}_n\|x_n - z\|^2\}$ consists of one point.*

Proof Define $\varphi(z) = \mathrm{LIM}_n\|x_n - z\|^2$ for each $z \in C$. Then the functionφ is convex and continuous on C, and $\varphi(z) \to \infty$ as $\|z\| \to \infty$. Set $r = \inf\{\varphi(z) : z \in C\}$. Since X is reflexive, it follows that there exists $u \in C$ such that $\varphi(u) = r$. Hence, D is nonempty. By Lemma 2.1, we infer that $u \in D$ if and only if

$$\mathrm{LIM}_n \langle z-u, J(x_n-u)\rangle \le 0 \tag{2.1}$$

for all $z \in C$.

Now we claim that D consists of one point. Indeed, let $u, v \in D$ and $u \neq v$. Then, by [4, Theorem 1], there exists $\delta > 0$ such that

$$\langle v-u, J(x_n-u)-J(x_n-v)\rangle = \langle x_n-u-(x_n-v), J(x_n-u)-J(x_n-v)\rangle \ge \delta > 0$$

for each $n \ge 1$, which hence implies that $\mathrm{LIM}_n\langle v-u, J(x_n-u)-J(x_n-v)\rangle \ge \delta > 0$.

But, it follows from (2.1) that for $u, v \in D$, $\mathrm{LIM}_n\langle v-u, J(x_n-u)\rangle \le 0$, $\mathrm{LIM}_n\langle u-v, J(x_n-v)\rangle \le 0$. Thus, we have $\mathrm{LIM}_n\langle v-u, J(x_n-u)-J(x_n-v)\rangle \le 0$. This arrives at a contradiction, which hence implies that $u=v$.

Lemma 2.3[11,Theorem 1] *Suppose X is a Banach space with uniformly normal structure, C is a nonempty bounded subset of X, and $T: C \to C$ is a uniformly L-Lipschitzian mapping with $L < N(X)^{1/2}$. Suppose also there exists a nonempty bounded closed convex subset E of C with the following property (P):*

$$\text{(P)} \quad x \in E \Rightarrow \omega_w(x) \subset E,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set

$$\{y \in X : y = \text{weak-}\lim_{j\to\infty} T^{n_j}x \ \text{ for some } n_j \uparrow \infty\}.$$

Then T has a fixed point in E.

3. Main results

Suppose C is a nonempty bounded closed convex subset of a Banach space X and $T: C \to C$ is uniformly L-Lipschitzian with $L \ge 1$ and asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$. Fix a u in C and define for each integer $n \ge 1$ the contraction $S_n: C \to C$ by

$$S_n(x) = (1-\frac{t_n}{Lk_n})u + \frac{t_n}{Lk_n}T^n x, \tag{II}$$

where $\{t_n\} \subset [0,1)$. Then the Banach Contraction Principle yields a unique point x_n fixed by S_n. Now the question naturally gives rise to whether the sequence $\{x_n\}$ converges strongly to a fixed point of T. The following is an answer.

Theorem 3.1 *Let X be a uniformly convex Banach space with a uniformly Gateaux differentiable norm,Cbe a nonempty bounded closed convex subset of X, and $T: C \to C$ be uniformly L-Lipschitzian with $1 \le L < N(X)^{1/2}$ and asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$. Let $\{t_n\} \subset [0,1)$ be chosen so that $\lim_{n\to\infty}(k_n-1)/(Lk_n-t_n) = 0$. Suppose in addition there holds the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, where the sequence$\{x_n\}$is generated by (II). Then,* *(1)* $D=\{x\in C:\varphi(x)=\min_{z\in C}\varphi(z)\}$ *is a singleton, say* $\{z_0\}$, *where* $\varphi(z)=\mathrm{LIM}_n\|x_n-z\|^2$;

(2) the following statements are equivalent:

(i) $z_0\in F(T)$;

(ii) $\|x_n-T^mz_0\|^2\le\langle x_n-T^mz_0,J(x_n-z_0)\rangle$ *for all* $m,n\ge 1$;

(iii) $\{x_n\}$ *converges strongly to* z_0.

Proof (1) Now, let LIM be a Banach limit and define $\varphi:C\to[0,\infty)$ by $\varphi(z)=\mathrm{LIM}_n\|x_n-z\|^2$. Since φ is continuous and convex, $\varphi(z)\to\infty$ as $\|z\|\to\infty$, and X is reflexive, φ attains its infimum over C. Hence, the set $D=\{x\in C:\varphi(x)=\min_{z\in C}\varphi(z)\}$ is nonempty, closed and convex. By Lemma 2.2, D consists of one point. Write $D=\{z_0\}$.

(2) At first, we show that (i)⇔(ii). Indeed, if z_0 is a fixed point of T in C, then it is easy to see that for all $m,n\ge 1$, $\|x_n-T^mz_0\|^2=\|x_n-z_0\|^2=\langle x_n-z_0,J(x_n-z_0)\rangle=\langle x_n-T^mz_0,J(x_n-z_0)\rangle$.

Conversely, suppose that for all $m,n\ge1$, $\|x_n-T^mz_0\|^2\le\langle x_n-T^mz_0,J(x_n-z_0)\rangle$.

Then we assert that z_0 is a fixed point of T in C. Indeed, let $y=w-\lim_{j\to\infty}T^{m_j}z_0$ be an arbitrary point of the weak ω-limit set $\omega_w(z_0)$ of T at z_0. Then from the weak lower semicontinuity of φ, the condition $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, and (3.1), we obtain

$$\begin{aligned}\varphi(y)\le&\liminf_{j\to\infty}\varphi(T^{m_j}z_0)\le\limsup_{m\to\infty}\varphi(T^mz_0)\\
=&\limsup_{m\to\infty}(\mathrm{LIM}_n\|x_n-T^mz_0\|^2)\\
\le&\limsup_{m\to\infty}(\mathrm{LIM}_n\langle x_n-T^mz_0,J(x_n-z_0)\rangle)\\
=&\limsup_{m\to\infty}(\mathrm{LIM}_n\langle(x_n-Tx_n)+(Tx_n-T^2x_n)+\cdots+\\
&(T^mx_n-T^mz_0),J(x_n-z_0)\rangle)\\
\le&\limsup_{m\to\infty}(\mathrm{LIM}_n\{\|x_n-Tx_n\|+L\|x_n-Tx_n\|+\cdots+\\
&L\|x_n-Tx_n\|\}d+\mathrm{LIM}_nk_m\|x_n-z_0\|^2)\\
=&\varphi(z_0)=\min_{z\in C}\varphi(z),\end{aligned}$$

where $d=\mathrm{diam}C$. Thus, by the definition of D, we have $y\in D=\{z_0\}$, which implies that $y=z_0$. This shows that $\omega_w(z_0)=\{z_0\}$, and hence $D=\{z_0\}$ satisfies the property (P). It follows from Lemma 2.3 that z_0 is a fixed point of T in C.

Secondly, we show that (i)⇔(iii). Indeed, if $\{x_n\}$ converges strongly to z_0, then, according to the condition $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, the point z_0 is a fixed point of T in C.

Conversely, suppose that z_0 is a fixed point of T in C. Then $F(T)$ is nonempty. Observe that for each $v\in F(T)$,

$$\begin{aligned}\langle x_n-T^nx_n,J(x_n-v)\rangle&=\langle x_n-v,J(x_n-v)\rangle+\langle v-T^nx_n,J(x_n-v)\rangle\\
&=\|x_n-v\|^2-\langle T^nx_n-v,J(x_n-v)\rangle\\
&\ge\|x_n-v\|^2-k_n\|x_n-v\|^2\ge-(k_n-1)d^2.\end{aligned}\tag{3.2}$$

Since x_n is a fixed point of S_n, it follows that $x_n - T^n x_n = \frac{Lk_n - t_n}{t_n}(u - x_n)$ and thus from the inequality (3.2) above, we obtain

$$\langle x_n - u, J(x_n - v)\rangle \leq s_n d^2, \tag{3.3}$$

where $s_n = t_n(k_n - 1)/Lk_n - t_n) \to 0$ as $n \to \infty$.

Since z_0 is a unique minimizer of φ over C, it follows from Lemma 2.1 that

$$\mathrm{LIM}_n\langle x - z_0, J(x_n - z_0)\rangle \leq 0$$

for all $x \in C$; in particular, we have

$$\mathrm{LIM}_n\langle u - z_0, J(x_n - z_0)\rangle \leq 0. \tag{3.4}$$

Combining (3.3) with (3.4), we get

$$\mathrm{LIM}_n\langle x_n - z_0, J(x_n - z_0)\rangle = \mathrm{LIM}_n\|x_n - z_0\|^2 \leq 0.$$

Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z_0. To complete the proof, suppose there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ which converges strongly to (say) y. Then y is a fixed point of T by the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Thus, it follows from (3.3) that $\langle z_0 - u, J(z_0 - y)\rangle \leq 0$ and $\langle y - u, J(y - z_0)\rangle \leq 0$.

Adding these two inequalities yields

$$\langle z_0 - y, J(z_0 - y)\rangle = \|z_0 - y\|^2 = 0$$

and hence $z_0 = y$. This implies the strong convergence of $\{x_n\}$ to z_0.

Theorem 3.2 *Let X be a uniformly smooth Banach space, C be a nonempty bounded closed convex subset of X, and $T : C \to C$ be uniformly L-Lipschitzian with $1 \leq L < N(X)^{1/2}$ and asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$. Let $\{t_n\}$ be a sequence in $[0,1)$ such that $\lim_{n\to\infty}(k_n - 1)/(Lk_n - t_n) = 0$. Suppose in addition there holds the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, where the sequence $\{x_n\}$ is generated by (II). Then,*

(1) $D = \{x \in C : \varphi(x) = \min_{z\in C}\varphi(z)\}$ is nonempty, closed and convex, where

$$\varphi(z) = \mathrm{LIM}_n\|x_n - z\|^2;$$

(2) if for each $x \in D$,

$$\|x_n - T^m x\|^2 \leq \langle x_n - T^m x, J(x_n - x)\rangle \quad \text{for all } m, n \geq 1, \tag{3.5}$$

then $D \cap F(T)$ is a singleton, say $\{z_0\}$, and $\{x_n\}$ converges strongly to z_0.

Proof (1) Let LIM be a Banach limit and define $\varphi : C \to [0,\infty)$ by $\varphi(z) = \mathrm{LIM}_n\|x_n - z\|^2$. Since φ is continuous and convex, $\varphi(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive, so, φ attains its infimum over C. Hence, the set $D = \{x \in C : \varphi(x) = \min_{z\in C}\varphi(z)\}$ is nonempty, closed and convex.

(2) Though D is not necessarily invariant under T, it does have the property (P). As a matter of fact, if x lies in D and $y = w - \lim_{j\to\infty} T^{m_j}x$ belongs to the weak ω-limit set $\omega_w(x)$ of T at x, then from the weak lower semicontinuity of φ, the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, and (3.5), we obtain

$$\begin{aligned}
\varphi(y) &\leq \liminf_{j\to\infty} \varphi(T^{m_j}x) \leq \limsup_{m\to\infty} \varphi(T^m x) \\
&= \limsup_{m\to\infty}(\mathrm{LIM}_n\|x_n - T^m x\|^2) \\
&\leq \limsup_{m\to\infty}(\mathrm{LIM}_n\langle x_n - T^m x, J(x_n - x)\rangle) \\
&= \limsup_{m\to\infty}(\mathrm{LIM}_n\langle (x_n - Tx_n) + (Tx_n - T^2x_n) + \cdots + \\
&\quad (T^m x_n - T^m x), J(x_n - x)\rangle) \\
&\leq \limsup_{m\to\infty}(\mathrm{LIM}_n\{\|x_n - Tx_n\| + L\|x_n - Tx_n\| + \cdots + \\
&\quad L\|x_n - Tx_n\|\}d + \mathrm{LIM}_n k_m\|x_n - x\|^2) \\
&= \varphi(x) = \min_{z\in C}\varphi(z),
\end{aligned}$$

where $d = \mathrm{diam}C$. Thus, by the definition of D, we have $y \in D$. This shows that $\omega_w(x) \subset D$, and hence D satisfies the property (P). It follows from Lemma 2.3 that T has a fixed point in D. Therefore, we know that $D \cap F(T) \neq \emptyset$.

On the other hand, for each $p \in F(T)$, we have

$$\begin{aligned}
\langle x_n - T^n x_n, J(x_n - p)\rangle &= \langle x_n - p, J(x_n - p)\rangle + \langle p - T^n x_n, J(x_n - p)\rangle \\
&= \|x_n - p\|^2 - \langle T^n x_n - p, J(x_n - p)\rangle \\
&\geq \|x_n - p\|^2 - k_n\|x_n - p\|^2 \\
&\geq -(k_n - 1)d^2.
\end{aligned}$$

Since x_n is a fixed point of S_n, it follows that $x_n - T^n x_n = \frac{Lk_n - t_n}{t_n}(u - x_n)$ and thus, we have

$$\langle x_n - u, J(x_n - p)\rangle \leq s_n d^2,$$

where $s_n = t_n(k_n - 1)/(Lk_n - t_n) \to 0$ as $n \to \infty$.

Let z be an arbitrary point in $D \cap F(T)$. Since z is a minimizer of φ over C, it follows that

$$\lim_{t\to 0_+} \frac{\varphi(z + t(x - z)) - \varphi(z)}{t} \geq 0$$

for all $x \in C$. This, together with the uniform smoothness of X, easily implies that (see [14, pages 46–47] for details) $\mathrm{LIM}_n\langle x - z, J(x_n - z)\rangle \leq 0$ for all $x \in C$; in particular, we have

$$\mathrm{LIM}_n\langle u - z, J(x_n - z)\rangle \leq 0. \tag{3.7}$$

Combining (3.6) with (3.7), we get

$$\mathrm{LIM}_n\|x_n - z\|^2 = \mathrm{LIM}_n\langle x_n - z, J(x_n - z)\rangle \leq 0.$$

Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z. To complete the proof, suppose there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ which converges strongly to (say) y. Then y is a fixed point of T by the condition $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Thus, it follows from (3.6) that $\langle z-u, J(z-y)\rangle \le 0$ and $\langle y-u, J(y-z)\rangle \le 0$. Adding these two inequalities yields $\langle z-y, J(z-y)\rangle = \|z-y\|^2 = 0$ and thus $z=y$. This implies the strong convergence of $\{x_n\}$ to z. By using the uniqueness of the limit, we conclude that $D\cap F(T)$ is a singleton, say $\{z_0\}$, and $\{x_n\}$ converges strongly to z_0.

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关于渐近伪压缩映象迭代逼近的表征

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摘要: 设 C 是 Banach 空间 X 的非空有界闭凸子集，$T:C\to C$ 既是一致 L-Lipschitz 映象，$L\ge 1$，又是渐近伪压缩映象，具有序列 $\{k_n\}\subset[1,\infty)$，$\lim_{n\to\infty}k_n=1$. 固定 $u\in C$. 对每个 $n\ge 1$，x_n 是压缩映象 $S_n(x)=(1-\frac{t_n}{Lk_n})u+\frac{t_n}{Lk_n}T^nx, \forall x\in C$ 的唯一不动点，其中，$\{t_n\}\subset[0,1)$. 在适当的条件下，本文表征了序列 $\{x_n\}$ 强收敛到 T 的不动点.

关键词: 不动点；渐近伪压缩映象；一致 Lipschitz 映象；一致正规结构； Banach 压缩原理.