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S-integral and Gronwall's Inequality

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Abstract: The S-integral is a generalized integral of Riemann type which is defined in terms of the Thomson's local systems. In this note we prove Gronwall-Bellman's inequality for the S-integral. As special cases we also obtain Gronwall-Bellman's inequalities for the Henstock integral and the Burkill approximately continuous integral.

Key words: S-integral; Gronwall-Bellman's inequality.
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1. Introduction

Thomson[7] studied the continuity and differentiation of a real function by means of what he calls local systems. In fact, local systems can also be used in the study of integration. In [11] such an integral, called the S-integral, was defined. The S-integral is a generalized integral of Riemann type which includes as special cases the Henstock integral[4], the Burkill approximately continuous integral[2] and an integral based on the dyadic derivative[4].

The classical Gronwall's inequality, which plays an important role in analysis of differential equations, is usually formulated and proved within continuous functions[3]. In [6], Schwabik proved the Gronwall's inequality for the Henstock integral which is used in the study of Kurzweil equations. In [9] and [10], we obtained the Gronwall's inequalities for the Burkill approximately continuous integral and the S-integral respectively. In this note we would like to prove a more general inequality of Gronwall type for the S-integral.

The paper is organized as follows. In §2, we recall some necessary notions and facts in the S-integral. In §3, we establish an integration by parts formula for the S-integral. In the last section, by using the results obtained in §3, we prove the Gronwall-Bellman's inequality for the S-integral, which is our main result. As special cases, we also give the Gronwall-Bellman's inequalities for the Henstock integral and the Burkill approximately continuous integral.

2. S-integral

In this section we briefly recall some necessary notions and facts in the S-integral. For details see [11].

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Let $\mathbf{R}$ be the real line and $2^{\mathbf{R}}$ the collection of all subsets of $\mathbf{R}$. Suppose for every $x \in \mathbf{R}$, there corresponds a nonempty $S(x) \subset 2^{\mathbf{R}}$ such that

(i) $\{x\} \notin S(x)$;

(ii) if $\sigma \in S(x)$, then $x \in \sigma$;

(iii) if $\sigma_1 \in S(x)$ and $\sigma_1 \subset \sigma_2$, then $\sigma_2 \in S(x)$;

(iv) if $\sigma \in S(x)$ and $\delta > 0$, then $\sigma \cap (x-\delta, x+\delta) \in S(x)$.

Then $S = \{S(x); x \in \mathbf{R}\}$ is called a local system[7].

Using the local system S one can define the S-limit or S-continuity of a function at a point x. Moreover, the S-derivative of f at x is defined to be

$$S\text{-D}f(x) = S\text{-}\lim_{y \to x} \frac{f(y)-f(x)}{y-x}. \tag{2.1}$$

If S-D$f(x)$ exists for each x in the domain, then f is said to be S- differentiable (see [7] for details).

A local system S is said to be bilateral if for every $x \in \mathbf{R}$, σ contains points on either side of x whenever σ is in $S(x)$. It is said to be filtering if for every $x \in \mathbf{R}$, we have $\sigma_1 \cap \sigma_2 \in S(x)$ whenever σ_1 and σ_2 belong to $S(x)$. It satisfies the intersection condition if for every collection of sets $\{\sigma(x); x \in \mathbf{R}\}$ with $\sigma(x) \in S(x)$ there is a positive function δ defined on $\mathbf{R}$ such that if $0 < y - x < \min\{\delta(x), \delta(y)\}$, then $\sigma(x) \cap \sigma(y) \cap [x, y] \neq \emptyset$.

Henceforth we will always assume that all local systems we use are bilateral, filtering and satisfy the intersection condition.

Let $S = \{S(x); x \in \mathbf{R}\}$ be a given local system. A collection of sets $\eta = \{\eta_x; x \in \mathbf{R}\}$ with $\eta_x \in S(x)$ is called a choice from S. For a choice η from S and a compact interval $[a, b]$, by an η-fine partition of $[a, b]$ we mean a finite collection

$$\{([x_{i-1}, x_i], t_i); i = 1, 2, \cdots, n\}$$

with the following properties

$$a = x_0 < x_1 < \cdots < x_n = b$$

and

$$t_i \in [x_{i-1}, x_i], \quad x_{i-1}, x_i \in \eta_{t_i}, \quad i = 1, 2, \cdots, n.$$

A collection $\{([u_i, v_i], t_i); i = 1, 2, \cdots, p\}$ is called an η-fine partial partition of $[a, b]$ if it has the following conditions

$$a \le u_1 \le v_1 \le u_2 \le v_2 \le \cdots \le u_p \le v_p \le b$$

and

$$t_i \in [u_i, v_i], \quad u_i, v_i \in \eta_{t_i}, \quad i = 1, 2, \cdots, p.$$

Definition 2.1[11] *A function $f : [a, b] \longrightarrow \mathbf{R}$ will be termed S-integrable if there is a number I such that for every $\varepsilon > 0$ there exists a choice η from the local system S such that if $\{([x_{i-1}, x_i], t_i); i = 1, 2, \cdots, n\}$ is an η-fine partition of $[a, b]$, then*

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - I\right| < \varepsilon. \tag{2.2}$$

The number I, written as $(S)\int_a^b f(t)\mathrm{d}t$, is called the S-integral of f.

The S-integral includes as special cases many known integrals of generalized Riemann type and shares many of their properties. The next theorem gives a descriptive definition of the S-integral.

A function $F:[a,b]\longrightarrow \mathbf{R}$ is said to satisfy the strong Lusin condition on $[a,b]$ with respect to the system S if for every $E\subset[a,b]$ of measure zero and every $\varepsilon>0$, there exists a choice η from S such that if $\{([u_i,v_i],t_i);i=1,2,\cdots,p\}$ is an η-fine partial partition of $[a,b]$ with $\{t_i;i=1,2,\cdots,p\}\subset E$, then we have

$$\sum_{i=1}^{p}|F(v_i)-F(u_i)|<\varepsilon. \tag{2.3}$$

Theorem 2.2[11] *A function $f:[a,b]\longrightarrow\mathbf{R}$ is S-integrable on $[a,b]$ if and only if there exists a function $F:[a,b]\longrightarrow\mathbf{R}$ satisfying the strong Lusin condition on $[a,b]$ with respect to S and S-$DF(x)=f(x)$ almost everywhere in $[a,b]$. In this case,*

$$(S)\int_a^b f(x)\mathrm{d}x=F(b)-F(a). \tag{2.4}$$

3. Integration by parts for the S-integral

In this section we establish an integration by parts formula for the S-integral, which will be used in the proof of our main result. Let S be a given local system and $[a,b]$ a compact interval.

Proposition 3.1 *Let $F:[a,b]\longrightarrow\mathbf{R}$ be a function that satisfies the strong Lusin condition on $[a,b]$ with respect to S. Then, for every function $G:[a,b]\longrightarrow\mathbf{R}$, every $E\subset[a,b]$ of measure zero and every $\varepsilon>0$, there exists a choice η from S such that if $\{([u_i,v_i],t_i);i=1,2,\cdots,p\}$ is an η-fine partial partition of $[a,b]$ with $\{t_i;i=1,2,\cdots,p\}\subset E$, then we have*

$$\sum_{i=1}^{p}\big|G(t_i)[F(v_i)-F(u_i)]\big|<\varepsilon. \tag{3.1}$$

Proof Put

$$E_k=\{x;x\in E,k-1\le|G(x)|<k\},\quad k=1,2,\cdots.$$

It is easy to see that $E=\bigcup_{k=1}^{\infty}E_k$ and that $E_k\cap E_l=\emptyset$ for $k\ne l$. For each nonempty E_k, take a choice $\eta^{(k)}$ from S such that if $\{([u_i,v_i],t_i);i=1,2,\cdots,p\}$ is an $\eta^{(k)}$-fine partial partition of $[a,b]$ with $\{t_i;i=1,2,\cdots,p\}\subset E_k$, then

$$\sum_{i=1}^{p}|F(v_i)-F(u_i)|<\varepsilon/(k2^k).$$

Now construct a new choice η as follows. For $x\in\mathbf{R}\setminus E$, set $\eta_x=\mathbf{R}$; for $x\in E$, set $\eta_x=\eta_x^{(k)}$ if some E_k contains x (existence and uniqueness of such E_k is obvious). We see that if

$\{([u_i, v_i], t_i); i = 1, 2, \cdots, p\}$ is an η-fine partial partition of $[a, b]$ with $\{t_i; i = 1, 2, \cdots, p\} \subset E$, then we have

$$\begin{aligned}\sum_{i=1}^{p} \left|G(t_i)[F(v_i) - F(u_i)]\right| &= \sum_{k\in N} \sum_{t_i\in E_k} \left|G(t_i)[F(v_i) - F(u_i)]\right| \\ &\le \sum_{k\in N} k \sum_{t_i\in E_k} |F(v_i) - F(u_i)|] \\ &\le \sum_{k\in N} k\varepsilon/(k2^k) < \varepsilon,\end{aligned}$$

where $N = \{k; E_k \cap \{t_1, t_2, \cdots, t_p\} \neq \emptyset\}$.

Proposition 3.2 *If $F, G : [a, b] \longrightarrow \mathbf{R}$ satisfy the strong Lusin condition on $[a, b]$ with respect to the system S, then so does their product FG.*

Proof Let $E \subset [a, b]$ of measure zero and $\varepsilon > 0$ be given. Take a choice $\eta^{(1)}$ from S such that if $\{([u_i, v_i], t_i); i = 1, 2, \cdots, p\}$ is an $\eta^{(1)}$-fine partial partition of $[a, b]$ with $\{t_i; i = 1, 2, \cdots, p\} \subset E$, then we have

$$\max\left\{\sum_{i=1}^{p} \left|F(v_i) - F(u_i)\right|, \sum_{i=1}^{p} \left|G(v_i) - G(u_i)\right|\right\} < \min\{\varepsilon/4, 1\}.$$

In particular

$$\max\left\{\left|F(t_i) - F(u_i)\right|, \left|G(v_i) - G(t_i)\right|\right\} < \min\{\varepsilon/4, 1\}, \quad i = 1, 2, \cdots, p.$$

By Proposition 3.1, we can take another choice $\eta^{(2)}$ from S such that if $\{([u_i, v_i], t_i); i = 1, 2, \cdots, p\}$ is an $\eta^{(2)}$-fine partial partition of $[a, b]$ with $\{t_i; i = 1, 2, \cdots, p\} \subset E$, then we have

$$\max\left\{\sum_{i=1}^{p} \left|G(t_i)[F(v_i) - F(u_i)]\right|, \sum_{i=1}^{p} \left|F(t_i)[G(v_i) - G(u_i)]\right|\right\} < \varepsilon/4.$$

Now put $\eta_x = \eta_x^{(1)} \cap \eta_x^{(2)}$, for $x \in \mathbf{R}$. Obviously, $\eta = \{\eta_x; x \in \mathbf{R}\}$ is a choice. Let $\{([u_i, v_i], t_i); i = 1, 2, \cdots, p\}$ be an η-fine partial partition of $[a, b]$ with $\{t_i; i = 1, 2, \cdots, p\} \subset E$. Then we see that

$$\begin{aligned}&\sum_{i=1}^{p} \left|F(v_i)G(v_i) - F(u_i)G(u_i)]\right| \\ &\le \sum_{i=1}^{p} \left|G(v_i)[F(v_i) - F(u_i)]\right| + \sum_{i=1}^{p} \left|F(u_i)[G(v_i) - G(u_i)]\right| \\ &\le \sum_{i=1}^{p} \left|G(v_i) - G(t_i)\right|\left|F(v_i) - F(u_i)\right| + \sum_{i=1}^{p} \left|G(t_i)[F(v_i) - F(u_i)]\right| + \\ &\quad \sum_{i=1}^{p} \left|F(t_i) - F(u_i)\right|\left|G(v_i) - G(u_i)\right| + \sum_{i=1}^{p} \left|F(t_i)[G(v_i) - G(u_i)]\right| \\ &\le \sum_{i=1}^{p} \left|F(v_i) - F(u_i)\right| + \sum_{i=1}^{p} \left|G(v_i) - G(u_i)\right| + \varepsilon/2 \\ &\le \varepsilon.\end{aligned}$$

This completes the proof.

Using Theorem 2.2 and Proposition 3.2, we can easily come to the next useful proposition.

Proposition 3.3 *Let $f, g : [a,b] \longrightarrow \mathbf{R}$ be S-integrable and*

$$F(x) = c_1 + (S)\int_a^x f(t)\mathrm{d}t, \quad G(x) = c_2 + (S)\int_a^x g(t)\mathrm{d}t,$$

where c_1, c_2 are constants. Then the function $fG + Fg$ is S-integrable on $[a,b]$, moreover

$$(S)\int_u^v [f(x)G(x) + F(x)g(x)]\mathrm{d}x = F(v)G(v) - F(u)G(u), \tag{3.2}$$

where $[u,v] \subset [a,b]$.

In the following we denote by $(H)\int_u^v \varphi(x)\mathrm{d}x$ the Henstock integral of a function $\varphi(x)$ which is Henstock integrable on the interval $[u,v]$. The next theorem gives an integration by parts formula for the S-integral.

Theorem 3.4 *Let $f : [a,b] \longrightarrow \mathbf{R}$ be S-integrable and $g : [a,b] \longrightarrow \mathbf{R}$ a function of bounded variation. Define*

$$F(x) = c_1 + (S)\int_a^x f(t)\mathrm{d}t, \quad G(x) = c_2 + (H)\int_a^x g(t)\mathrm{d}t,$$

where c_1, c_2 are constants. Assume that F is Henstock integrable on $[a,b]$. Then the product fG is S-integrable on $[a,b]$, moreover,

$$(S)\int_u^v f(x)G(x)\mathrm{d}x = F(v)G(v) - F(u)G(u) - (H)\int_u^v F(x)g(x)\mathrm{d}x, \tag{3.3}$$

where $[u,v] \subset [a,b]$.

Proof g is Henstock integrable on $[a,b]$ since it is bounded variation on $[a,b]$. Hence g is also S-integrable on $[a,b]$ and

$$G(x) = c_2 + (H)\int_a^x g(t)\mathrm{d}t = c_2 + (S)\int_a^x g(t)\mathrm{d}t. \tag{3.4}$$

Therefore, $fG + Fg$ is S-integrable on $[a,b]$.

On the other hand, by a result in [5], we know that the product Fg is Henstock integrable on $[a,b]$ since F is Henstock integrable on $[a,b]$. Hence Fg is also S-integrable on $[a,b]$ and

$$(S)\int_u^v F(x)g(x)\mathrm{d}x = (H)\int_u^v F(x)g(x)\mathrm{d}x, \quad [u,v] \subset [a,b]. \tag{3.5}$$

This together with the S-integrability of $fG + Fg$ implies that the product fG is S-integrable on $[a,b]$. (3.3) easily follows from (3.2) and (3.5).

4. Gronwall-Bellman's inequality for S-integral

We now prove Gronwall-Bellman's inequality for the S-integral, which is our main result. Let S be a given local system and $T > 0$. In additon, for a function $\varphi(t)$, its Burkill approximately continuous integral is denoted by $(AP)\int_a^b \varphi(t)\mathrm{d}t$.

Theorem 4.1 *Let $f,\ g:[0,T]\longrightarrow \mathbf{R}$ be S-integrable. Assume that $G(x)=(S)\int_a^x g(t)\mathrm{d}t$ is Henstock integrable on $[0,T]$. If there is a constant $k>0$ such that*

$$f(x)\le g(x)+k\cdot(S)\int_0^x f(t)\mathrm{d}t,\quad x\in[0,T], \tag{4.1}$$

then

$$f(x)\le g(x)+k\cdot(S)\int_0^x g(t)e^{k(x-t)}\mathrm{d}t,\quad x\in[0,T]. \tag{4.2}$$

If (4.1) holds a.e., then so does (4.2).

Proof Define

$$F(x)=(S)\int_0^x f(t)\mathrm{d}t,\quad x\in[0,T]. \tag{4.3}$$

Then (4.1) becomes

$$f(x)\le g(x)+kF(x),\quad x\in[0,T]. \tag{4.4}$$

Put $\varphi(x)=-ke^{-kx}$, $\Phi(x)=e^{-kx}$. Then we see

$$S\text{-}\mathrm{D}\Phi(x)=\varphi(x),\quad x\in[0,T]. \tag{4.5}$$

By the Theorem 2.2, $\varphi(x)$ is S-integrable on $[0,T]$ and

$$\Phi(x)=1+(S)\int_0^x \varphi(t)\mathrm{d}t,\quad x\in[0,T]. \tag{4.6}$$

Multiplying both sides of (4.4) by $\Phi(x)$ yields

$$f(x)\Phi(x)+F(x)\varphi(x)\le g(x)\Phi(x),\quad x\in[0,T]. \tag{4.7}$$

By Proposition 3.3 and Theorem 3.4, both $f(x)\Phi(x)+F(x)\varphi(x)$ and $g(x)\Phi(x)$ are S-integrable on $[0,T]$, and moreover,

$$F(x)\Phi(x)=(S)\int_0^x[f(t)\Phi(t)+F(t)\varphi(t)]\mathrm{d}t\le(S)\int_0^x g(t)\Phi(t)\mathrm{d}t,\quad x\in[0,T] \tag{4.8}$$

which implies

$$F(x)\le(S)\int_0^x g(t)\Phi(t)[\Phi(x)]^{-1}\mathrm{d}t=(S)\int_0^x g(t)e^{k(x-t)}\mathrm{d}t,\quad x\in[0,T]. \tag{4.9}$$

Combining (4.1), (4.3) and (4.9), we get (4.2). If (4.1) holds a.e., then in the same way we can show that (4.2) also holds a.e. The proof is complete.

Theorem 4.1 gives Gronwall-Bellman's inequality for the S-integral. As special cases of the theorem, we immediately get Gronwall-Bellman's inequalities for the Henstock integral and the Burkill approximately continuous integral as follows.

Corollary 4.2 *Let $f, g: [0,T] \longrightarrow \mathbf{R}$ be Henstock integrable. If there is a constant $k>0$ such that*

$$f(x) \le g(x) + k\cdot (H)\int_0^x f(t)\mathrm{d}t, \quad x\in[0,T], \tag{4.10}$$

then

$$f(x) \le g(x) + k\cdot (H)\int_0^x g(t)e^{k(x-t)}\mathrm{d}t, \quad x\in[0,T]. \tag{4.11}$$

If (4.10) holds a.e., then so does (4.11).

Corollary 4.3 *Let $f, g: [0,T] \longrightarrow \mathbf{R}$ be integrable in the sense of the Burkill approximately continuous integral. Assume that $G(x) = (AP)\int_a^x g(t)\mathrm{d}t$ is Henstock integrable on $[0,T]$. If there is a constant $k>0$ such that*

$$f(x) \le g(x) + k\cdot (AP)\int_0^x f(t)\mathrm{d}t, \quad x\in[0,T], \tag{4.12}$$

then

$$f(x) \le g(x) + k\cdot (AP)\int_0^x g(t)e^{k(x-t)}\mathrm{d}t, \quad x\in[0,T]. \tag{4.13}$$

If (4.12) holds a.e., then so does (4.13).

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S- 积分与 Gronwall 不等式

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摘要: S- 积分是利用 Thomson 的局部系定义的一种广义 Riemann 积分. 本文证明了针对 S-积分的 Gronwall-Bellman 不等式. 作为特例，我们也获得了针对 Henstock 积分和 Burkill 近似连续积分的 Gronwall-Bellman 不等式.

关键词: S- 积分； Gronwall-Bellman 不等式.