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Generalized Vector Quasi-Variational-Like Inequalities without Monotonity and Compactness

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Abstract: In this paper, some existence theorems of a solution for generalized vector quasi-variational-like inequalities without any monotonity conditions in a noncompact topological space setting are proven by the maximal element theorem.

Key words: set-valued mapping; generalized vector quasi-variational-like inequalities; noncompact topological space; L-η-condition; existence.
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1. Introduction

The Vector Variational Inequality (for short, VVI) in a finite dimensional Euclidean space was introduced in [1] and the applications were given. Chen and Cheng[2] studied the VVI in infinite dimensional space and applied it to Vector Optimization Problem (for short, VOP). Since then, many authors have intensively studied the VVI on different assumptions in infinite-dimensional spaces. Lee et al.[3], Lin et al.[4], Konnov and Yao[5], Yang and Yao[6], and Oettli and Schlager[7] studied the generalized vector variational inequality and obtained some existence results. Chen et al.[8] and Lee et al.[9] introduced and studied the generalized vector quasi-variational inequality and established some existence theorems. Ding[10,11] and Luo[12] studied the generalized vector variational-like inequalities. Ding[13] introduced and studied a class of generalized vector quasi-variational-like inequality problem (in short, GVQVLIP). By employing the scalarization technique, Ding[13] established several existence results for (GVQVLIP) involving C_+-η-monotone and weakly C_+-η-monotone set-valued mappings.

In this paper, we use the maximal element theorem with an escaping sequence in [20] to prove the existence results of a solution for (GVQVLIP) without any monotonity conditions in a noncompact topological space setting.

2. Preliminaries

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Let Y be a real Hausdorff topological vector space and X be a nonempty convex subset in a real locally convex Hausdorff topological vector space E. we denote $L(E,Y)$ the space of all continuous linear operators from E into Y and by $< u, y >$ the evaluation of $u \in L(E,Y)$ at $y \in E$. Let σ is the family of all bounded subsets of X whose union is total in E, i.e., the linear hull of $\cup\{S : S \in \sigma\}$ is dence in X. Let β be a neighbourhood base of 0 in Y. When S runs through σ, V through β, the family $M(S,V) = \{l \in L(E,Y) : \cup_{x\in S} < l, x > \subset V\}$ is a neighbourhood base of 0 in $L(E,Y)$ at $x \in E$ (see [14, pp. 79-80]). By the Corollary of Schaefer[14], $L(E,Y)$ becomes a locally convex topological vector space under σ-topology, where Y is assumed a locally convex topological space.

Lemma 2.1[10] *Let E and Y be real Hausdorff topological vector spaces and $L(E,Y)$ be the topological vector space under the σ-topology. Then, the bilinear mapping*

$$\langle \cdot, \cdot \rangle : L(E,Y) \times E \to Y$$

is continuous on $L(E,Y)$, where $< l, x >$ denotes the evaluation of the linear operator $l \in L(X,Y)$ at $x \in X$.

Let int$A$ and CoA denote the interior and convex hull of a set A, respectively, and $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with int$C(x) \neq \emptyset$ for each $x \in X$. Let $\eta : X \times X \to E$ be a single-valued mapping, $D : X \to 2^X$ and $T : X \to 2^{L(E,Y)}$ be two set-valued mappings. Ding[13] introduced a generalized vector quasi-variational-like inequality problem (GVQVLIP), which is to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}) : \quad \langle \hat{v}, \eta(y, \bar{x}) \rangle \notin -\text{int}C(\bar{x}). \tag{1}$$

Then the point $\bar{x}$ is said to be a solution of the (GVQVLIP).

It is easy to see that $\bar{x}$ is a solution of the (GVQVLIP) is equivalent to $\bar{x}$ in X satisfying $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \langle T(\bar{x}), \eta(y, \bar{x}) \rangle \not\subseteq -\text{int}C(\bar{x}), \tag{2}$$

where $\langle T(\bar{x}), \eta(y, \bar{x}) \rangle = \cup_{v \in T(\bar{x})} \langle v, \eta(y, \bar{x}) \rangle$.

The following problems are special cases of the (GVQVLIP).

(i) For all $x \in X$, if $D(x) \equiv X$, then the (GVQVLIP) reduces to the generalized vector variational-like inequality problem (in short, GVVLIP) which is to find $\bar{x}$ in X such that there exists an $\hat{v} \in T(\bar{x})$ satisfying

$$\langle \hat{v}, \eta(y, \bar{x}) \rangle \notin -\text{int}C(\bar{x}), \quad \forall y \in X. \tag{3}$$

This problem was studied in [10–12].

(ii) If T is a single-valued mapping and $\eta(y, x) = y - g(x), \forall x, y \in X$, where $g : X \to E$ is a single-valued mapping, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, satisfying

$$\langle T(\bar{x}), y - g(\bar{x}) \rangle \notin -\text{int}C(\bar{x}), \quad \forall y \in D(\bar{x}). \tag{4}$$

This is a new problem. If for all $x \in X$, $D(x) \equiv X$, then the problem (4) reduces to finding $\bar{x}$ in X such that

$$\langle T(\bar{x}), y - g(\bar{x})\rangle \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in X. \tag{5}$$

The problem (5) was considered by Siddiqi et al.[15].

(iii) If $\eta(y,x) = y - x$, $\forall x, y \in X$, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}): \quad \langle \hat{v}, y - x\rangle \notin -\mathrm{int}C(\bar{x}). \tag{6}$$

Problem (6) is called the generalized vector quasivariational inequality problem (GVQVIP) which is new. When $C(x) = C$, $\forall x \in X$ is a constant cone, problem (6) was studied by Chen and Li[8] and Lee et al.[9].

(iv) If $D(x) \equiv X$, $\forall x \in X$ and $\eta(y,x) = y - x$, $\forall x, y \in X$, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that

$$\forall y \in X, \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y, \bar{x})\rangle \notin -\mathrm{int}C(\bar{x}). \tag{7}$$

Problem (7) and its special cases are called the generalized vector variational inequality (GVVIP) which was introduced and studied in [3–7].

(v) If T is a single-valued function, then the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\langle T(\bar{x}), \eta(y, \bar{x})\rangle \notin -\mathrm{int}C(\bar{x}), \quad \forall y \in D(\bar{x}). \tag{8}$$

When $D(x) \equiv X$, $\forall x \in X$, problem (8) and its special cases were studied in [1,2].

(vi) If $Y = R$ and $C(x) = [0, \infty), \forall x \in X$, then $L(E, Y) = E^*$, where E^* is the dual space of E, and the (GVQVLIP) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}), \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y, \bar{x})\rangle \geq 0. \tag{9}$$

Problem (9) includes many classes of scalar type generalized quasivariational inequality and generalized variational-like inequality problems as special cases (see [16] and the references therein).

In order to prove the main results, we need the following definitions and lemmas.

Definition 2.1[10] *Let E, Y be two real topological vector spaces, X be a nonempty and convex subset of E, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone for each $x \in X$. Let $\eta : X \times X \to E$ be a single-valued mapping. $T : X \to 2^{L(E,Y)}$ is said to satisfy the generalized L-η-condition iff for any finite set $\{y_1, y_2, \cdots, y_n\}$ in X, $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ with $\alpha_j \geq 0$ and $\sum_{j=1}^n \alpha_j = 1$, there exists $\bar{v} \in T(\bar{x})$, such that*

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j \eta(y_j, \bar{x})\right\rangle \notin -\mathrm{int}C(\bar{x}).$$

Remark 2.1 If $\eta(y,x)$ is affine in the first argument and $\forall x \in X$, $\exists v \in T(x)$, such that

$$\langle \bar{v}, \eta(x, x)\rangle \notin -\mathrm{int}C(x),$$

then T satisfies the generalized L-η-condition.

Remark 2.2 If $\eta(y,x) = y - x$, $\forall x, y \in X$, then we have that

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j (y_j - \bar{x}) \right\rangle = \langle \bar{v}, \bar{x} - \bar{x} \rangle = 0 \notin -\mathrm{int}C(\bar{x}), \quad \forall v \in T(\bar{x}).$$

And hence T satisfies the generalized L-η-condition trivially.

Let X be a topological space. A subset S of X is said to be compactly open (respectively, compactly closed) in X if for any nonempty compact subset K of X, $S \cap K$ is open (respectively, closed) in S . Let Y be a topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then, T is said to be open valued if the set $T(x)$ is open in X for each $x \in X$. T is said to have open lower sections if T^{-1} is open valued, i.e., the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$. T is said to be compactly open valued if the set $T(x)$ is compactly open in X for each $x \in X$, and T is said to have compactly open lower sections if T^{-1} is compactly open valued. Clearly, each open-valued (respectively, closed-valued) mapping $T : X \to 2^Y$ is compactly open-valued (respectively, compactly closed-valued)[33],[40]. T is said to be upper semicontinuous if, for any $x_0 \in X$ and for each open set U in Y containing $T(x_0)$, there is a nerghborhood V of x_0 in X such that $T(x) \subseteq U$, for all $x \in V$; T is said to be closed, if the set $\{(x,y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.2[17] *Let X and Y be topological spaces. If $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with closed values, then T is closed.*

Lemma 2.3[19] *Let X and Y be topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 2.4[18] *Let X and Y be two topological spaces. Suppose $T : X \to 2^Y$ is a set-valued mapping having open lower sections, then the set-valued mapping $F : X \to 2^Y$ defined by, for each $x \in X$, $F(x) = \mathrm{Co}T(x)$ has open lower sections.*

Definition 2.2[20] *Let E be a topological space and X be a subset of X, such that $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing (in the sense that $X_n \subseteq X_{n+1}$) sequence of nonempty compact sets. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be an escaping sequence from X (relative to $\{X_n\}_{n=1}^{\infty}$) iff for each $n = 1, 2, \cdots, \exists m > 0$, such that $x_k \notin X_k$, $\forall k \geq m$.*

Lemma 2.5[20] *Let X be a subset of a topological vector space E such that $\{X_n\}_{n=1}^{\infty}$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets of X. Assume that the set-valued mapping $S : X \to 2^X$ satisfies the following conditions:*

(i) For each $x \in X$, $S^{-1}(x) \cap X_n$ is open in X_n for all $n = 1, 2, \cdots$;

(ii) For each $x \in X$, $x \notin \mathrm{Co}S(x)$;

(iii) For each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in X_n$ for all $n = 1, 2, \cdots$, which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, there exist $n \in N$ and $y_n \in X_n$ such that $y_n \in S(x_n) \cap X_n$.

Then there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

3. Existence results

In this section, we present some existence results of the (GVQVLIP) without any monotone conditions in a noncompact topological space setting.

Theorem 3.1 *Let E be a real locally convex Hausdorff topological vector space, let X be a subset of E such that $X = \cup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subset of X, let Y be a real Hausdorff topological vector space, and let $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be a set-valued mapping with nonempty convex values and compactly open lower sections, the set $W = \{x \in X : x \in D(x)\}$ be closed, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\mathrm{int}C(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-\mathrm{int}C) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta : X \times X \to E$ be continuous with respect to the second argument, such that T satisfies the generalized L-η-condition. Suppose that*

(A1) for each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in X_n$ for all $n = 1, 2, 3, \cdots$, which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, $\exists m \in N$, and $\exists z_m \in D(x_m) \cap X_m$ such that, $\forall s_m \in T(x_m)$, we have

$$\langle s_m, \eta(z_m, x_m)\rangle \in -\mathrm{int}C(x_m).$$

Then, the (GVQVLIP) has a solution $\bar{x} \in X$.

Proof Define a set-valued mapping $P : X \to 2^X$ by

$$\begin{aligned} P(x) &= \{y \in X : \langle T(x), \eta(y, x)\rangle \subseteq -\mathrm{int}C(x)\} \\ &= \{y \in X :< v, \eta(y, x) > \in -\mathrm{int}C(x), \forall v \in T(x)\}, \quad \forall x \in X. \end{aligned}$$

We first prove that $x \notin \mathrm{Co}P(x)$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \mathrm{Co}P(\bar{x})$. Then there exist finite points $y_1, y_2, \cdots, y_n$ in X, and $\alpha_j \geq 0$ with $\sum_{j=1}^{n} \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^{n} \alpha_j y_j$ and $y_j \in P(\bar{x})$ for all $j = 1, 2, \cdots, n$. That is, $\langle v, \eta(y_j, \bar{x}) > \in -\mathrm{int}C(\bar{x})$, $\forall v \in T(x)$ and $j = 1, 2, \cdots, n$. Since $\mathrm{int}C(\bar{x})$ is convex, we obtain

$$\langle v, \sum_{j=1}^{n} \alpha_j \eta(y_j, \bar{x})\rangle \in -\mathrm{int}C(\bar{x}), \quad \forall v \in T(x),$$

which contradicts the fact that T satisfies the generalized L-η-condition. Therefore, $x \notin \mathrm{Co}P(x)$ for all $x \in X$.

We also define a set-valued mapping $G : X \to 2^X$ by

$$G(x) = \begin{cases} D(x) \cap \mathrm{Co}P(x) & \text{if} \quad x \in W, \\ D(x) & \text{if} \quad x \in X \setminus W. \end{cases}$$

Then for each $x \in X$, $G(x)$ is convex. Suppose that there exists $\bar{x} \in X$ such that $\bar{x} \in G(\bar{x})$. If $\bar{x} \in W$, then $\bar{x} \in D(x) \cap \mathrm{Co}P(x)$, which contradicts $x \notin \mathrm{Co}P(x)$ for all $x \in X$. If $x \notin W$, then $G(\bar{x}) = D(\bar{x})$ which implies that $\bar{x} \in G(\bar{x})$, a contradiction. Hence for all $x \in X$, $x \notin G(x) = \mathrm{Co}G(x)$, the condition (ii) of Lemma 2.5 is satisfied.

Now we prove that the set

$$\begin{aligned} P^{-1}(y) &= \{x \in X :< T(x), \eta(y,x) >\subseteq -\mathrm{int}C(x)\} \\ &= \{x \in X : \langle v, \eta(y,x)\rangle \in -\mathrm{int}C(x), \quad \forall v \in T(x)\} \end{aligned}$$

is open for each $y \in X$. That is, P has open lower sections in X. Consider the set-valued mapping $S : X \to 2^Y$ defined by

$$\begin{aligned} S(y) &= \{x \in X : \langle T(x), \eta(y,x)\rangle \notin -\mathrm{int}C(x)\} \\ &= \{x \in X : \exists\ v \in T(x) \ \text{ such that } \ \langle v, \eta(y,x)\rangle \notin -\mathrm{int}C(x)\}. \end{aligned}$$

We only need to prove that $S(y)$ is closed for all $y \in X$. In fact, consider a net $x_t \in S(y)$ such that $x_t \to x \in X$. Since $x_t \in S(y)$, there exists $s_t \in T(x_t)$ such that $\langle s_t, \eta(y, x_t)\rangle \notin -\mathrm{int}C(x_t)$. From the upper semicontinuity and compact values of T and Lemma 2.3, it suffices to find a subset $\{s_{t_j}\}$ which converges to some $s \in T(x)$. By Lemma 2.1, we know that $\langle\cdot\rangle$ is continuous, and hence

$$\langle s_{t_j}, \eta(y, x_{t_j})\rangle \to \langle s, \eta(y,x)\rangle.$$

By Lemma 2.2 and upper semicontinuity of M, we have $\langle s, y-x\rangle \notin -\mathrm{int}C(x)$, and hence $x \in S(y)$, $S(y)$ is closed. Therefore, P has open lower sections in X.

By Lemma 2.4, $CoP^{-1}(y)$ is also open for each $y \in X$. Since $D^{-1}(y)$ is compactly open for each $y \in X$,

$$\begin{aligned} G^{-1}(y) &= \{x \in X : y \in G(x)\} \\ &= \{x \in W : y \in [D(x) \cap \mathrm{Co}P(x)]\} \cup \{x \in X \setminus W : y \in D(x)\} \\ &= \Big(W \cap D^{-1}(y) \cap \mathrm{Co}P^{-1}(y)\Big) \cup \Big[(X \setminus W) \cap D^{-1}(y)\Big] \\ &= \Big[\Big(W \cap D^{-1}(y) \cap \mathrm{Co}P^{-1}(y)\Big) \cup (X \setminus W)\Big] \cap \Big[\Big(W \cap D^{-1}(y) \cap \mathrm{Co}P^{-1}(y)\Big) \cup D^{-1}(y)\Big] \\ &= \Big\{X \cap \Big[\Big(D^{-1}(y) \cap \mathrm{Co}P^{-1}(y)\Big) \cup (X \setminus W)\Big]\Big\} \cap \Big[\Big(W \cup D^{-1}(y)\Big) \cap \Big(D^{-1}(y)\Big)\Big] \\ &= \Big[\Big(D^{-1}(y) \cap \mathrm{Co}P^{-1}(y)\Big) \cup (X \setminus W)\Big] \cap D^{-1}(y) \\ &= \Big(D^{-1}(y) \cap (\mathrm{Co}P^{-1}(y))\Big) \cup \Big((X \setminus W) \cap (D^{-1}(y))\Big). \end{aligned}$$

Therefore, $G^{-1}(y)$ also has compactly open values in X for all $y \in Y$, the condition (i) of Lemma 2.5 is satisfied. Condition (A1) implies condition (iii) of Lemma 2.5. Therefore, by Lemma 2.5, there exists $\bar{x} \in X$ such that $G(\bar{x}) = \emptyset$. Since for each $x \in X$, $D(x)$ is nonempty, we have $\bar{x} \in D(\bar{x})$ such that $D(\bar{x}) \cap \mathrm{Co}P(\bar{x}) = \emptyset$, which implies that $\bar{x} \in D(\bar{x})$ such that $D(\bar{x}) \cap P(\bar{x}) = \emptyset$, that is, $\bar{x} \in D(\bar{x})$, and $\forall y \in D(\bar{x})$, $\exists v \in T(\bar{x})$ satisfying $\langle v, \eta(y,\bar{x})\rangle \notin -\mathrm{int}C(\bar{x})$. That is, the (GVQVLIP) has a solution $\bar{x} \in X$.

By Theorem 3.1 and Remark 2.1, we have

Corollary 3.2 *Let E be a locally convex Hausdorff topological vector space , X be a subset of E such that $X=\cup_{n=1}^{\infty}X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subset of X, Y be a Hausdorff topological vector space, and $L(E,Y)$ be equipped with the σ-topology. Let $D: X\to 2^X$ be a set-valued mapping with nonempty convex values and compact open lower sections, the set $W=\{x\in X: x\in D(x)\}$ be closed, $C: X\to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\mathrm{int}C(x)\neq\emptyset$ for each $x\in X$, and the set-valued mapping $M=Y\backslash(-\mathrm{int}C): X\to 2^Y$ be upper semicontinuous on X. Let $T: X\to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta: X\times X\longrightarrow E$ be continuous with respect to the second argument and affine with respect to the first argument such that $\forall x\in X$, $\exists v\in T(x)$, satisfying*

$$\langle\bar{v},\eta(x,x)\rangle\notin-\mathrm{int}C(x).$$

Suppose that

(A1) for each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n\in X_n$ for all $n=1,2,3,\cdots$, which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, $\exists m\in N$, and $\exists z_m\in D(x_m)\cap X_m$ such that, $\forall s_m\in T(x_m)$, we have

$$\langle s_m,\eta(z_m,x_m)\rangle\in-\mathrm{int}C(x_m).$$

Then, the (GVQVLIP) has a solution $\bar{x}\in X$.

Remark 3.1 If $D(x)=X$ for all $x\in X$, then by Corollary 3.2, we recover Theorem 2 in [12]. Hence, both Theorem 3.1 and Corollary 3.2 are generalizations of Theorem 1 and Theorem 2 in [12].

Theorem 3.3 *Let E be a locally convex Hausdorff topological vector space , X be a subset of E such that $X=\cup_{n=1}^{\infty}X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subset of X, Y be a Hausdorff topological vector space, and $L(E,Y)$ be equipped with the σ-topology. Let $D: X\to 2^X$ be a set-valued mapping with nonempty convex values and compact open lower sections, the set $W=\{x\in X: x\in D(x)\}$ be closed, $C: X\to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\mathrm{int}C(x)\neq\emptyset$ for each $x\in X$, and the set-valued mapping $M=Y\backslash(-\mathrm{int}C): X\to 2^Y$ be upper semicontinuous on X. Let $T: X\to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta: X\times X\to E$ be continuous with respect to the second argument. Suppose that there exists a mapping h : $X\times X\to Y$, such that:*

(i) $\forall x,y\in X$, $\exists v\in T(x)$, such that

$$h(x,y)-\langle v,\eta(y,x)\rangle\in-\mathrm{int}C(x);$$

(ii) For any finite set $\{y_1,y_2,\cdots,y_n\}\subseteq X$ and $\bar{x}=\sum_{j=1}^{n}\alpha_jy_j$ with $\alpha_j\geq 0$ and $\sum_{j=1}^{n}\alpha_j=1$, there is a $j\in\{1,2,\cdots,n\}$, such that $h(\bar{x},y_j)\notin-\mathrm{int}C(\bar{x})$.

(iii) For each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in X_n$ for all $n = 1, 2, 3, \cdots$, which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, $\exists m \in N$, and $\exists z_m \in D(x_m) \cap X_m$ such that, $\forall s_m \in T(x_m)$, we have

$$\langle s_m, \eta(z_m, x_m)\rangle \in -\mathrm{int}C(x_m).$$

Then, the (GVQVLIP) has a solution $\bar{x} \in X$.

Proof Define two set-valued mappings $P : X \to 2^X$, $P_1 : X \to 2^X$ by

$$P(x) = \{y \in X : \langle v, \eta(y, x)\rangle \in -\mathrm{int}C(x), \quad \forall v \in T(x)\}, \forall x \in X.$$

$$P_1(x) = \{y \in X : h(x, y) \in -\mathrm{int}C(x)\}, \quad \forall x \in X.$$

We first prove that $x \notin \mathrm{Co}(P_1(x))$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \mathrm{Co}(P_1(\bar{x}))$. Then there exist finite points $y_1, y_2, \cdots, y_n$ in X, and $\alpha_j \geq 0$ with $\sum_{j=1}^{n} \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^{n} \alpha_j y_j$ and $y_j \in P_1(\bar{x})$ for all $j = 1, 2, \cdots, n$. That is, $h(\bar{x}, y_j) \in -\mathrm{int}C(\bar{x})$, $j = 1, 2, \cdots, n$. This contradicts to the condition (ii). Therefore $x \notin \mathrm{Co}(P_1(x))$ for all $x \in X$.

The Condition (i) implies that $P_1(x) \supseteq P(x)$ for all $x \in X$. Hence, $x \notin \mathrm{Co}(P(x))$, $\forall x \in X$.

The remainder of the proof is the same as that in the proof of Theorem 3.1.

Corollary 3.4 *Let E be a locally convex Hausdorff topological vector space , X be a subset of E such that $X = \cup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subset of X, Y be a Hausdorff topological vector space, and $L(E, Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be a set-valued mapping with nonempty convex values and compact open lower sections, the set $W = \{x \in X : x \in D(x)\}$ be closed, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $intC(x) \neq \emptyset$ for each $x \in X$, and the set-valued mapping $M = Y \backslash (-intC) : X \to 2^Y$ be upper semicontinuous on X. Let $T : X \to 2^{L(E,Y)}$ be upper semicontinous on X with compact values and $\eta : X \times X \to E$ be continuous with respect to the second argument. Suppose that there exists a mapping $h : X \times X \to Y$, such that:*

(i) $\forall x, y \in X$, $h(x, y) - \langle T(x), \eta(y, x)\rangle \in -\mathrm{int}C(x)$;

(ii) The set $\{y \in X : h(x, y) \in -\mathrm{int}C(x)\}$ is convex for all $x \in X$;

(iii) $h(x, x) \notin -\mathrm{int}C(x), \forall x \in X$;

(iv) For each sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \in X_n$ for all $n = 1, 2, 3, \cdots$, which is escaping from X relative to $\{X_n\}_{n=1}^{\infty}$, $\exists m \in N$, and $\exists z_m \in D(x_m) \cap X_m$ such that, $\forall s_m \in T(x_m)$, we have

$$\langle s_m, \eta(z_m, x_m)\rangle \in -\mathrm{int}C(x_m).$$

Then, the (GVQVLIP) has a solution $\bar{x} \in X$.

Proof From the proof of Corollary 3 in [10], we know that the Conditions (ii) and (iii) imply the Condition (ii) of Theorem 3.3. Then, by Theorem 3.3, we know the conclusion holds.

Remark 3.2 Theorem 3.1, Corollary 3.2, Theorem 3.3 and Corollary 3.4, respectively, generalize the main results in [10] from the cases of generalized set-valued variational-like inequalities to the cases of generalized set-valued quasi-variational-like inequalities.

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没有单调性和紧性的广义向量拟似变分不等式

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摘要: 本文利用极大元定理在非紧拓扑空间设置下证明了没有单调性的广义向量拟似变分不等式解的存在性定理.

关键词: 集值映射；广义向量拟似变分不等式；非紧拓扑空间； L-η- 条件；存在性.