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Influence of s-Semipermutability of Some Subgroups of Prime Power Order on Structure of Finite Groups

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Abstract: A subgroup H of a finite group G is called *semipermutable* if it is permutable with every subgroup K of G with $(|H|,|K|)=1$, and s-semipermutable if it is permutable with every Sylow p-subgroup of G with $(p,|H|)=1$. In this paper, we investigate the influence of $s-$semipermutablity of some subgroups of prime power order of a finite group on its supersolvablility.

Key words: s-semipernutable subgroups; Sylow tower; supersolvable groups
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1. Introduction

All groups considered in this paper will be finite.

Recall that two subgroups H and K of a group G are said to be permute if $HK=KH$. A subgroup of a group G is called quasinormal in G if it permutes with every subgroup of G. A subgroup of G is called s-quasinormal in G if it permutes with every Sylow subgroup of G. A subgroup H of G is called s-semipermutable if H permutes with every Sylow p-subgroup of G with $(p,|H|)=1$. It is easy to see that a s-quasinormal subgroup of a group G is a s-semipermutable subgroup of G. The converse is not true in general. For example, a Sylow 3-subgroup of the symmetric group S_4 of degree 4 is s-semipermutable in S_4 but not s-quasinormal in S_4.

Several authors have investigated the structure of a finite group when some subgroups of prime order of the group are well-situated in the group. Itô[4] proved that a finite group G of odd order is nilpotent provided that all minimal subgroups of G lie in the center of G. Buckley[2] proved that if all minimal subgroups of an odd order group are normal, then the group is supersolvable. Shaalan[8] proved that if every subgroup of G of prime order or 4 is s-quasinormal in G, then G is supersolvable. Recently, M.Asaad and M.Ramadan[1,6,7] proved the following: Put $\pi(G)=\{p_1,\cdots,p_n\}$, where $p_1>\cdots>p_n$. Let P_i be a Sylow p_i-subgroup of G and let the exponent of $\Omega(P_i)$ be $p_i^{e_i}$, where $i=1,\cdots,n$. Suppose that all members of the family $\{H|H\le\Omega(P_i),\mathrm{Exp}H=p_i^{e_i},i=1,\cdots,n\}$ are normal (quasinormal, s-quasinormal) in G.

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Then G is supersolvable. In this paper, we obtained the same conclusion if s-quasinormality is replaced by s-semipermutability. Obviously, s-semipermutability is a weaker concept than that of s-quasinormality. Hence, our results can be regarded as a generalization of that of M.Ramadan. It should be pointed out that the argument in this paper is different from that of [7]. Our notation is standard and taken mainly from [3].

2. Preliminaries

We will give some lemmas that are useful to the proofs of the theorems.

Lemma 2.1 *Let p be the smallest prime dividing $|G|$ and P be an abelian p-subgroup of G of exponent p^e. If each subgroup of P of exponent p^e is s-semipermutable in G, then $P \leq N_G(Q)$, where $Q \in \mathrm{Syl}_q(G)$, $p \neq q$.*

Proof Since P is an abelian p-subgroup, we can suppose that $P = \langle x_1 \rangle \times \cdots \times \langle x_s \rangle$, where $o(x_1) = p^e \geq \cdots \geq o(x_s)$. It is clear that $P = \langle x_1, x_2 \cdots, x_s \rangle = \langle x_1, x_1x_2, \cdots, x_1x_s \rangle$. Let $y_1 = x_1$, $y_i = x_1x_i$, $i = 2, \cdots, s$. Since P is an abelian subgroup of exponent p^e and $o(x_1) = p^e$, it follows that $o(y_i) = p^e$, $i = 1, \cdots, s$. By hypothesis, $\langle y_i \rangle Q = Q\langle y_i \rangle$, for all $Q \in \mathrm{Syl}_q(G)$, where $p \neq q$, $i = 1, \cdots, s$. Since p is the smallest prime dividing $|G|$, we have that $y_i \in N_G(Q)$, $i = 1, \cdots, s$, by [9, II, Th 5.5]. Therefore $P \leq N_G(Q)$. The proof is complete. □

Lemma 2.2 *Let G be a finite group. If P is a subnormal and s-semipermutable p-subgroup of G, where p is a prime, then P is s-quasinormal in G.*

Proof First we prove $P \leq O_p(G)$. Since P is subnormal in G, there exists a series $P = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$. Since $P \lhd H_1$, we have that $P \leq O_p(H_1)$. On the other hand, since $O_p(H_1)$ char $H_1 \lhd H_2$, it follows that $O_p(H_1) \leq O_p(H_2)$. Hence $P \leq O_p(H_2)$. By the same argument, we have that $P \leq O_p(H_n) = O_p(G)$.

Let Q be a q-Sylow subgroup. If $p = q$, then $PQ = QP = Q$ since $P \leq O_p(G) \leq Q$. If $P \neq Q$, then $PQ = QP$ since P is s-semipermutable in G. Therefore, P is an s-quasinormal subgroup of G. The lemma is proved. □

Lemma 2.3 *Let p be the smallest prime dividing $|G|$ and P be an abelian normal Sylow p-subgroup of G of exponent p^e. If each subgroup of P of exponent p^e is s-semipermutable in G, then $G = P \times K$ where K is a p'-Hall subgroup of G.*

Proof By Schur-Zassenhaus Theorem, there exists a p'-Hall subgroup K of G such that $G = P \rtimes K$. It follows from Lemma 2.1 that $P \leq N_G(K)$. Consequently, $G = P \times K$. □

As an immediate consequence of Lemma 2.2 and [7, Theorem 2.7], we have the following

Lemma 2.4 *Let P be a normal p-subgroup of G of exponent p^e such that G/P is supersolvable, where $e \geq 1$. Suppose that all member of the family $\{H | H \leq P, H' = 1, \mathrm{Exp}H = p^e\}$ are s-semipermutable in G. Then G is supersolvable.* □

For a finite p-group P, we write

$$\Omega(P) = \begin{cases} \Omega_1(P) & \text{if} \quad p > 2, \\ \Omega_2(P) & \text{if} \quad p = 2, \end{cases}$$

where $\Omega_i(P) = \langle x \in P \mid o(x) | p^i \rangle$.

Lemma 2.5 *Let P be a normal p-subgroup of G such that G/P is supersolvable and let the exponent of $\Omega(P)$ be p^e, where $e \geq 1$. Suppose that all member of the family $\{H | H \leq \Omega(P), H' = 1, \mathrm{Exp}H = p^e\}$ are s-semipermutable in G. Then G is supersolvable.*

Proof This is an immediate result of Lemma 2.2 and [7, Lemma 2.12]. □

3. Main results

Theorem 3.1 *Let p be the smallest prime dividing $|G|$ and P be a Sylow p-subgroup of G of exponent p^e, where $e \geq 1$. Suppose that all members of the family $\{H | H \leq P, H' = 1, \mathrm{Exp}H = p^e\}$ are s-semipermutable in G. Then G has a normal p-complement.*

Proof We prove the theorem by induction on $|G|$. Since $H^g Q = QH^g$ for all $Q \in \mathrm{Syl}_q(G)$, $q \neq p$, H^g is s-semipermutable in G, where H is an abelian subgroup of P of exponent p^e, $g \in G$. Let $N = \langle H^g | H \leq P, H' = 1, \mathrm{Exp}H = p^e, g \in G \rangle$. Then $N \trianglelefteq G$.

If $N = G$, then $Q \trianglelefteq G$ by Lemma 2.1, where $Q \in \mathrm{Syl}_q(G)$, $q \neq p$. Hence G has a normal p-complement. So we can assume $N < G$.

If N is not a p-group, then N has a nontrivial normal p-complement K by induction on $|G|$. So $K \trianglelefteq G$ since K char $N \trianglelefteq G$. We now consider the quotient group G/K. Since K is a p'-subgroup of G, G/K satisfies the hypothesis. Hence G/K has a normal p-complement L/K by induction on $|G|$. Since K is a p'-subgroup, L is a normal p-complement of G.

Assume N is a p-group. Let H be an abelian subgroup of N of exponent p^e of maximal order. Then H is s-semipermutable in G by hypothesis. It follows that HQ is a subgroup of G, where Q is a Sylow q-subgroup of G, and $p \neq q$. Since N is a normal p-subgroup of G, it follows that H is a subnormal subgroup of G. Therefore H is a subnormal Hall subgroup of HQ. Thus H is normal in HQ . On the other hand, Q is normal in HQ by Lemma 2.1. So $HQ = H \times Q$. In particular, $Q \leq C_G(H)$. So $O^p(G) \leq C_G(H)$. If $C_G(H) < G$, then $C_G(H)$ has a normal p-complement by induction on $|G|$. Hence $O^p(G)$ has a normal p-complement and also does G. Thus we may assume that $C_G(H) = G$. Then $H \leq Z(G)$. Now our choice of H implies that $H = N$ and $N \leq Z(G)$. So $N\langle x \rangle$ is an abelian subgroup of P of exponent p^e, where x is an element of P. By the definition of N, we can obtain that $N\langle x \rangle = N$. Hence $P = N$. Applying Lemma 2.3, we conclude that G has a normal p-complement. Thus completes the proof. □

Corollary 3.1 *Put $\pi(G) = \{p_1, p_2, \cdots, p_n\}$ where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G of exponent $p_i^{e_i}$, where $i = 2, \cdots, n$. Suppose that all members of the family $\{H | H \leq P_i, H' = 1, \mathrm{Exp}H = p_i^{e_i}, i = 2, \cdots, n\}$ are s-semipermutable in G. Then G possesses an ordered Sylow tower.*

Proof By Theorem 3.1, G has a normal p_n−complement. Assume that K is the normal p_n-complement of P_n in G. By induction, K possesses an ordered Sylow tower. Therefore, G possesses an ordered Sylow tower. □

Next, we give one of the main results:

Theorem 3.2 *Put $\pi(G) = \{p_1, p_2, \cdots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G of exponent $p_i^{e_i}$, where $i = 1, \cdots, n$. Suppose that all members of the family $\{H|H \leq P_i, H' = 1, ExpH = p_i^{e_i}, i = 1, \cdots, n\}$ are s-semipermutable in G. Then G is supersolvable.*

Proof By Corollary 3.1, G possesses an ordered Sylow tower. Then P_1 is normal in G. By Schur-Zassenhaus theorem, G has a p_1'-Hall subgroup K which is a complement to P_1 in G. Hence K is supersolvable by induction on $|G|$. Now, it follows from Lemma 2.4 that G is supersolvable. □

Theorem 3.3 *Let K be a normal subgroup of G such that G/K is supersolvable. Put $\pi(K) = \{p_1, p_2, \cdots, p_s\}$, where $p_1 > p_2 > \cdots > p_s$ and let P_i be a Sylow p_i-subgroup of K of exponent $p_i^{e_i}$, where $i = 1, \cdots, s$. Suppose that all members of the family $\{H|H \leq P_i, H' = 1, \mathrm{Exp}H = p_i^{e_i}, i = 1, \cdots, s\}$ are s-semipermutable in G. Then G is supersolvable.*

Proof We prove the theorem by induction on $|G|$. Theorem 3.2 implies that K is supersolvable and so P_1 is normal in K. Hence P_1 is normal in G since P_1 is a Sylow p_1-subgroup of K. Also $(G/P_1)/(K/P_1) \cong G/K$ is supersolvable. Now we conclude that G/P_1 is supersolvable by induction on $|G|$. Now, it follows from Lemma 2.4 that G is supersolvable. The theorem is proved.

We are now to prove the following results:

Theorem 3.4 *Let p be the smallest prime dividing $|G|$, P be a Sylow p-subgroup of G and the exponent of $\Omega(P)$ be p^e, where $e \geq 1$. Suppose that all members of the family $\mathcal{H} = \{H|H \leq \Omega(P), H' = 1, \mathrm{Exp}H = p^e\}$ are s-semipermutable in G. Then G has a normal p-complement.*

Proof We prove the theorem by induction on $|G|$. Since $H^gQ = QH^g$ for all $Q \in \mathrm{Syl}_q(G)$, $q \neq p$, H^g is s-semipermutable in G, where $H \in \mathcal{H}$. Let $N = \langle H^g | H \in \mathcal{H},\ g \in G\rangle$. Then $N \trianglelefteq G$. It follows that $N \leq N_G(Q)$ Since $H^g \leq N_G(Q)$ by Lemma 2.1, where $H \in \mathcal{H}$, $g \in G$ $Q \in \mathrm{Syl}_q(G)$, $p \neq q$.

If $N = G$, then $Q \trianglelefteq G$, $\forall Q \in \mathrm{Syl}_q(G)$, $p \neq q$. Hence G has a normal p-complement. So we may assume $N < G$.

If N is not a p-group, then N has a nontrivial normal p-complement K by induction on $|G|$. Clearly, $K \trianglelefteq G$. We now consider the quotient group G/K. Since K is a p'-subgroup of G, and $PK/K \cong P/P \cap K = P$, we have that G/K satisfies the hypothesis. Hence G/K has a normal p-complement L/K by induction on $|G|$. It follows that L is a normal p-complement of G.

If N is a p-group, then H is subnormal in G, $\forall H \in \mathcal{H}$. By Lemma 2.1, we have that

$HQ = H \times Q$, $\forall Q \in \mathrm{Syl}_q(G)$, $q \neq p$. Let $L = HO^p(G)$. It follows that $H \leq Z(L)$ and $L_p = H(P \cap O^p(G))$ is a Sylow p-subgroup of L. Take any element x of L_p, where $o(x) = p$ or $o(x) = 4$ when $p = 2$. Then $H\langle x\rangle \in \mathcal{H}$ and $H\langle x\rangle$ is a subnormal subgroup of G. Similarly, we have that $H\langle x\rangle \leq Z(H\langle x\rangle O^p(G)) = Z(L)$ and $x \in Z(L)$. By [9, IX, Th 6.1], it follows that L is p-nilpotent. Therefore, $O^p(G)$ has a normal p-complement K and K is a normal p-complement of G. This completes the proof. □

Corollary 3.2 *Put $\pi(G) = \{p_1, p_2, \cdots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G and let the exponent of $\Omega(P_i)$ be $p_i^{e_i}$, where $i = 2, \cdots, n$. Suppose that all members of the family $\{H | H \leq \Omega(P_i), H' = 1, \mathrm{Exp}H = p_i^{e_i}, i = 2, \cdots, n\}$ are s-semipermutable in G. Then G possesses an ordered Sylow tower.*

Proof By Theorem 3.4, G has a normal p_n-complement. Assume that K is the normal p_n-complement of P_n in G. By induction, K possesses an ordered Sylow tower. Therefore, G possesses an ordered Sylow tower. □

Theorem 3.5 *Put $\pi(G) = \{p_1, p_2, \cdots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G and let the exponent of $\Omega(P_i)$ be $p_i^{e_i}$, where $i = 1, \cdots, n$. Suppose that all members of the family $\{H | H \leq P_i, H' = 1, \mathrm{Exp}H = p_i^{e_i}, i = 1, \cdots, n\}$ are s-semipermutable in G. Then G is supersolvable.*

Proof We prove the theorem by induction on $|G|$. By Corollary 3.2, we have that G possesses an ordered Sylow tower. Then P_1 is normal in G. By Schur-Zassenhaus theorem, G has a p_1'-Hall subgroup K which is a complement to P_1 in G. Hence K is supersolvable by induction on $|G|$. Now, it follows from Lemma 2.5 that G is supersolvable. □

Theorem 3.6 *Let K be a normal subgroup of G such that G/K is supersolvable. Put $\pi(K) = \{p_1, p_2, \cdots, p_s\}$, where $p_1 > p_2 > \cdots > p_s$. Let P_i be a Sylow p_i-subgroup of K and the exponent of $\Omega(P_i)$ be $p_i^{e_i}$, where $i = 1, \cdots, s$. Suppose that all members of the family $\{H | H \leq \Omega(P_i), H' = 1, \mathrm{Exp}H = p_i^{e_i}, i = 1, \cdots, s\}$ are s-semipermutable in G. Then G is supersolvable.*

Proof We prove the theorem by induction on $|G|$. Theorem 3.5 implies that K is supersolvable and so P_1 is normal in K. Hence P_1 is normal in G. Also, $(G/P_1)/(K/P_1) \cong G/K$ is supersolvable. Now we conclude that G/P_1 is supersolvable by induction on $|G|$. Now, it follows from Lemma 2.5 that G is supersolvable. The theorem is proved. □

Remark Our results are more general than that of Ramadan. For example, let $G = S_3$, the Symmetric group of degree three. Obviously, G is supersolvable. Since a Sylow 2-subgroup G_2 of G is not s-quasinormal in G, G does not satisfy the conditions of Ramadan [7, Th 2.6, Th 2.13]. Therefore we cannot obtain the supersolvability of G by [7, Th 2.6, Th 2.13]. On the other hand, since G_2 is s-semipermutable in G, by our Theorem 3.2 or Theorem 3.5, we can get that G is supersolvable.

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素数幂阶子群的 s- 半置换性对有限群结构的影响

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摘要: 设 G 为有限群，G 的一个子群 H 称为半置换的，若对任意的 $K \leq G$, 只要 $(|K|, |H|) = 1$, 就有 $KH = HK$; H 称为 s– 半置换的，若对任意的 $p||G|$, 只要 $(p, |H|) = 1$, 就有 $PH = HP$, 其中 $P \in \mathrm{Syl}_p(G)$. 本文考察了素数幂阶子群的 s- 半置换性对有限群的超可解性的影响.

关键词: s- 半置换子群； Sylow 塔；超可解群.