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Some Properties of duo QF-1 Rings

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Abstract: It is proved that a Noetherian duo right QF-1 ring is a QF-ring. And some results of linearly compact duo QF-1 rings are investigated.

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Throughout this paper, rings are always associative with identity and modules are unitary. We freely use the terminology and notation of [1].

A ring R is called a right QF-1 ring in case every faithful right R-module is balanced, that is, there is a canonical ring isomorphism from R to $\mathrm{Biend}(M_R)$ for every faithful right R-module M. Many properties of QF-1 rings were presented by Thrall[2] and Camillo[3]. R is a right PF-ring in case every faithful right R-module is a generator. According to Faith[4], each generator is balanced, thus a right PF-ring is always a right QF-1 ring. The converse is not true when R is non-commutative. But whether a commutative QF-1 ring is PF is still open. Dickson&Fuller[5] and Camillo[3] proved that a commutative artinian QF-1 ring is QF, respectively. Ringel[6] and Storrer[7] generalized it to the commutative Noetherian case.

In this paper, we prove that a Noetherian duo right QF-1 ring is QF, which generalizes Ringel's result[6]. At the same time, we investigate linearly compact duo QF-1 rings and duo self-injective QF-1 rings.

We denote $r_R(X)$ the right annihilator of X in R, and J the Jacobson radical of R. Let $\mathrm{Soc}(M)$ be the socle of module M, $E(M)$ the injective hull of M and $\mathrm{Rad}(M)$ the Jacobson radical of M.

A ring R is called duo in case each one-side ideal is two-sided. Obviously, $Ra = aR$ for each a in a duo ring R.

Lemma 1 *Let R be a duo Noetherian ring with simple essential socle. Then $l_R(\mathrm{Soc}R)$ is nilpotent.*

Proof Since $\mathrm{Soc}R$ is simple, $l_R(\mathrm{Soc}R)$ is a maximal ideal of R. Let $N = l_R(\mathrm{Soc}R)$, $N \supseteq N^2 \supseteq N^3 \supseteq \cdots$, then $r_R(N) \subseteq r_R(N^2) \subseteq r_R(N^3) \subseteq \cdots$, and there is an n such that $r_R(N^n) = r_R(N^{n+1})$ since R is Noetherian. If $N^{n+1} \neq 0$, let $K = \{a \in N | N^n a \neq 0\}$, $K \neq \emptyset$, then

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$\{r_R(a)|a \in K\}$ has a maximal element. Suppose $r_R(a)$ is the maximal element. For each nonzero $b \in N, r_R(b)$ is an essential ideal, so $r_R(b) \cap aR \neq 0$. There is an $r_0 \in R$ such that $ar_0 \neq 0$ and $bar_0 = 0$, that is, $r_0 \notin r_R(a)$ but $r_0 \in r_R(ba)$. We have $ba \in N$ and $r_R(a) \subset r_R(ba)$. By the maximality of $r_R(a)$, we get $N^n ba = 0$. Since b is an arbitrary element in N, $N^{n+1}a = 0$. But $a \in r_R(N^{n+1}) = r_R(N^n)$, that is, $N^n a = 0$, a contradiction. Thus $N^{n+1} = 0$.

Lemma 2 *Let R be a duo Noetherian ring with simple essential socle. Then R is a local QF-ring.*

Proof By Lemma 1 and [1, Corollary 15.10], the maximal ideal $l_R(\mathrm{Soc}R) \subseteq J$, hence $J = l_R(\mathrm{Soc}R)$ and R is local. By Lemma 1 and [1, Theorem 15.20] again, R is semiprimary, so R is an Artinian ring. Now by [1, Corollary 31.8] R is a QF-ring.

The following two lemmas are very important in this paper. The idea of their proofs, given below for completion, comes from [6, Lemmas 3 and 4].

Lemma 3 *Let R be a local duo right QF-1 ring. If J is finitely generated, then R has non-zero socle.*

Proof Suppose $\mathrm{Soc}R = 0$, we have $r_R(J) = 0$, then J is faithful. Since J is finitely generated, let $J_R = x_1R + x_2R + \cdots + x_kR$. Then R-homomorphism $\varphi : R \to R^k$ defined by $r \mapsto (x_1r, x_2r, \cdots, x_kr)$ is a monomorphism. And $\mathrm{Im}\varphi \subseteq J^k = \mathrm{Rad}(R^k)$. Set $M_1 = R, M_2 = M_1^k, \varphi_1 = \varphi : M_1 \to M_2$ and $M_{n+1} = M_n^k, \varphi_n = \varphi_{n-1}^k : M_n \to M_{n+1}$. Then all φ_n's are monic and $\mathrm{Im}\varphi_n \subseteq \mathrm{Rad}(M_{n+1})$. Let M be the direct limit of the diagram

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} \cdots.$$

Since all φ_n's are monic, we may assume that each $M_n \subseteq M$ and φ_n is the inclusion map. So $M = \cup_{n=1}^{\infty} M_n$ and the socle of M is zero since the socle of each M_n is zero. Assume X is a maximal submodule of M. Then $M_n \nsubseteq X$ for some n. If $m \in M_n \backslash X$, then $(X \cap M_{n+1}) + mR = M_{n+1} \cap (X + mR) = M_{n+1} \cap M = M_{n+1}$. Since $m = \varphi_n(m) \in \mathrm{Rad}(M_{n+1}), mR$ is superfluous in M_{n+1}. Then $X \cap M_{n+1} = M_{n+1}, M_n \subseteq M_{n+1} \subseteq X$, a contradiction. So M has no maximal submodules. But M is faithful, according to [3, Lemma 2], this contradicts to the QF-1 ring assumption of R.

Lemma 4 *Let R be a local duo right QF-1 ring with non-zero socle. Then* Soc$R$ *is simple and essential.*

Proof Let S be a minimal ideal of R. We show that each nonzero ideal contains S.

Assume A is a proper ideal of R such that $S \cap A = 0$. Take $0 \neq s \in S$ and $0 \neq a \in A$. We consider the module $M_R = R^2/(s, a)R$. Since R^2 is projective, for every $\gamma \in \mathrm{End}(M)$, the following diagram

$$\begin{array}{ccc} R^2 & \xrightarrow{\bar{\gamma}} & R^2 \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\gamma} & M \end{array}$$

commutes and $\bar{\gamma}$ is lifted by γ, consequently $\bar{\gamma}$ takes $(s,a)R$ into $(s,a)R$, where $\pi : R^2 \to M$ is the natural epimorphism. And the operation of $\bar{\gamma}$ on R^2 is just that of some matrix

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \quad r_{ij} \in R.$$

So $(s,a)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (sr_{11}+ar_{21}, sr_{12}+ar_{22}) = (rs, ra)$ for some $r \in R$. We note that $Ra = aR$ for each a in a duo ring R, then $ar_{21} = rs - sr_{11} \in Ra \cap Rs = 0$. So r_{21} is not invertible, but R is local, $r_{21} \in J$. Similarly, $sr_{12} = ra - ar_{22} \in Ra \cap Rs = 0$ implies that $r_{12} \in J$. Define an additive homomorphism f of R^2 into itself by $(r_1, r_2) \to (0, sr_2)$. Since $s \in S \subseteq \mathrm{Soc}R$ and $(s,a)R \subseteq J^2$, f maps $(s,a)R$ into 0 and therefore induces an additive endomorphism f of M. Let $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathrm{End}(M_R) = T$ and $\overline{(r_1, r_2)} \in M$. Then $sr_2r_{21} = 0 = sr_1r_{12}$ since $r_{12}, r_{21} \in J$. Hence

$$[f\overline{(r_1,r_2)}]\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \overline{(0,sr_2)}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \overline{(sr_2r_{21}, sr_2r_{22})} = \overline{(0, sr_1r_{12}+sr_2r_{22})}$$

$$= f\overline{(r_1r_{11}+r_2r_{21}, r_1r_{12}+r_2r_{22})} = f\left[\overline{(r_1,r_2)}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}\right].$$

That is, $f \in \mathrm{End}({}_TM) = \mathrm{Biend}(M_R)$. Next we show M is faithful. If $Mr = 0$ then $(1,1)r \in (s,a)R, r \in sR \cap aR = 0$. Thus $r = 0$ and M is faithful. Since R is right QF-1, M_R is balanced. There is an $r_0 \in R$ such that $f(m) = mr_0$ for each $m \in M$. Since $f\overline{(0,1)} = \overline{(0,s)} \neq 0, r_0 \neq 0$. And $0 = f\overline{(1,0)} = r_0\overline{(1,0)} = \overline{(r_0,0)}$, hence $(r_0, 0) \in (s,a)R$. Let $r_0 = r_1s$ and $0 = r_1a$. Since R is local,$r_1 \in J$, then $r_0 = r_1s = 0$, a contradiction.

Theorem 5 *Let R be a duo Noetherian ring. If R is a right QF-1 ring, then R is QF.*

Proof From the proof of [6, Lemma 2], we note that a duo Noetherian ring also has the property of no non-zero maps between the injective hulls of two non-isomorphic simple R-modules. By [3, Theorem 7], we may assume R is local. Now by Lemmas 3, 4 and 2, R is QF.

Now we investigate linearly compact rings. A module M is called linearly compact [8, Section 3] in case any finitely solvable congruence $m \equiv m_i (\mathrm{mod}\ M_i)$ is solvable, where M_i's are submodules of M and $m_i \in M$. A ring R is right (left) linearly compact if the regular module R_R (${}_RR$) is linearly compact. Since each one-sided ideal in a duo ring is an ideal, a duo ring is right linearly compact iff it is left linearly compact , so we simply speak of a linearly compact duo ring. We recall that a ring R is right PF if R_R is an injective cogenerator.

Theorem 6 *Let R be a linearly compact duo right QF-1 ring. If J is finitely generated, then R is a PF-ring.*

Proof Since a linearly compact ring is semiperfect and each idempotent lies in the center of a duo ring, a linearly compact duo ring is a finite product of local linearly compact duo rings. We suppose R is a local ring. Using Lemma 3, R has a nontrivial socle. And $\mathrm{Soc}R$ is simple and essential by Lemma 4. Now by [9, Lemma 3.2] R is a (two-sided) PF-ring.

Theorem 7 *Let R be a linearly compact duo right QF-1 ring. If $\cap_{n=1}^{\infty} J^n$ is finitely generated, then R is a QF-ring.*

Proof Let R be local. Since R is linearly compact, J/J^2 is finitely generated semisimple. Assume $J = \sum_{i=1}^{m} x_i R + J^2$, then

$$J^2 = J \cdot (\sum_{i=1}^{m} x_i R + J^2) \subseteq \sum_{i=1}^{m} x_i R + J^3,$$

so

$$J = \sum_{i=1}^{m} x_i R + J^2 \subseteq \sum_{i=1}^{m} x_i R + J^3 \subseteq \sum_{i=1}^{m} x_i R + J^2 = J.$$

We have $J = \sum_{i=1}^{m} x_i R + J^3$. By the similar method, $J = \sum_{i=1}^{m} x_i R + J^n$ for each n. By [8, Corollary 3.9],

$$J = \bigcap_{n=1}^{\infty} (\sum_{i=1}^{m} x_i R + J^n) = \sum_{i=1}^{m} x_i R + \bigcap_{n=1}^{\infty} J^n.$$

J is finitely generated since $\cap_{n=1}^{\infty} J^n$ is finitely generated. Thus R is a PF-ring by Theorem 6. And now it follows from [8, Corollary 17.5 and Lemma 17.1] that $\cap_{n=1}^{\infty} J^n = 0$ and hence R is a Noetherian ring. Hence R is QF.

Recall that a ring R is right $(FPF)PF$ if every (finitely generated) faithful right R-module is a generator. We now discuss right duo right self-injective right QF-1 rings. XIN Lin[12] shows that a left self-injective left duo and left QF-1 ring is a left PF-ring, if J is nil and J/J^2 is finite generated. Here we give another condition for this result. From [10] a right self-injective ring is duo iff it is right duo. We simply assume R is a duo right self-injective ring.

Lemma 8 *If R is duo right self-injective, then R is a right FPF-ring.*

Proof Let $M = m_1 R + m_2 R + \cdots + m_n R$ be a finitely generated faithful right R-module. Then

$$0 = r_R(\sum_{i=1}^{n} m_i R) = \bigcap_{i=1}^{n} r_R(m_i R) \subseteq \bigcap_{i=1}^{n} r_R(m_i).$$

If $\cap_{i=1}^{n} r_R(m_i) \neq 0$, then $\cap_{i=1}^{n} r_R(m_i)$ is a non-zero two-sided ideal since R is a duo ring. We have

$$M \cdot \bigcap_{i=1}^{n} r_R(m_i) = \sum_{i=1}^{n} (m_i R \cdot \bigcap_{i=1}^{n} r_R(m_i)) = 0.$$

But M is faithful, a contradiction. Therefore $\cap_{i=1}^{n} r_R(m_i) = 0$. Define $\varphi : R \to M^n, r \mapsto (m_1 r, m_2 r, \cdots, m_n r)$, φ is monic. Since R is right self-injective, $M^n = R \oplus X$. Then M is a generator, so R is right FPF.

Lemma 9 *Let R be a local duo right self-injective ring. If $E(R/J)$ is finitely generated as right R-module, then R is a right PF-ring.*

Proof Since R is local, $E(R/J)$ is a minimal (injective) cogenerator, which is a faithful right

R-module. By Lemma 8 $E(R/J)$ is a generator. Since R_R is finitely generated projective, $E(R/J) \to R_R \to 0$ spilts for some n. That is, R_R is a direct summand of $E(R/J)^n$, which is finitely cogenerated. Hence R_R is finitely cogenerated. Now by [11, Proposition 24.32(d)], R is a right PF-ring.

Corollary 10 *Let R be a local duo right self-injective ring. If there is a finitely generated cogenerator as right R-module, then R is a right PF-ring.*

Proposition 11 *Let R be a duo right self-injective and right QF-1 ring. If there is a finitely generated cogenerator as right R-module, then R is a right PF-ring.*

Proof By [12, Lemma 3] and [3, Theorem 7], we may assume R is local. By Corollary 10, R is a right PF-ring.

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关于 duo QF-1 环的若干性质

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摘要: 证明了 Noetherian duo 右 QF-1 环是 QF 环, 并给出了线性紧 duo 右 QF-1 环的几个结论

关键词: QF 环; QF-1 环; duo 环.