第25卷第3期 数学研究与评论 Vol.25, No.3
2005年8月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Aug., 2005

Article ID: 1000-341X(2005)03-0447-06 **Document code:** A

A Note on Lattice Paths with Diagonal Steps in Three-Dimensional Space

LU Qing-lin

(Dept. of Math., Xuzhou Normal University, Jiangsu 221116, China)
(E-mail: qllu@xznu.edu.cn)

Abstract: Jr. Stocks[4] discussed lattice paths from $(0,0,0)$ to (n,n,n) with diagonal steps under some restrictions. In this note, we give simpler formulas for the main results in [4], and extend them to a general case.

Key words: lattice path; diagonal step; Andre's reflection principle.
MSC(2000): 05A15
CLC number: O157

1. Introduction

Unrestricted and restricted (minimal) lattice paths in two-dimensional and three-dimensional coordinate spaces have been widely discussed in the literature[1–5]. Unrestricted lattice paths in the plane with diagonal steps yield the Delannoy numbers[6]. Restricted lattice paths in the plane, which have only horizontal and vertical steps, and never rise above the plane diagonal, yield the well-documented Catalan numbers.

Moser and Zayachkowski[2] discussed the lattice paths from $(0,0)$ to (n,n) with diagonal steps never rise above the line $y=x$. Jr. Stocks[4] extended the results of [2] to three-dimensional space.

Consider lattice paths from $(0,0,0)$ to (m,n,k) where the possible moves are of four types: (i) a x-increasing step, e.g. $(x,y,z)\to(x+1,y,z)$, (ii) a y-increasing step, e.g. $(x,y,z)\to(x,y+1,z)$, (iii) a z-increasing step, e.g. $(x,y,z)\to(x,y,z+1)$, and (iv) a diagonal step, e.g. $(x,y,z)\to(x+1,y+1,z+1)$. Let

$f(m,n,k)$= total number of paths from $(0,0,0)$ to (m,n,k),

$Q(n)$= number of paths from $(0,0,0)$ to (n,n,n) whose lattice points are on the $(n,0,0)$ side of the plane $y=x$, except for the points $(0,0,0)$ and (n,n,n), and

$Q'(n)$= number of paths from $(0,0,0)$ to (n,n,n) whose lattice points are on the $(n,0,0)$ side of the plane $y=x$, or on the plane $y=x$.

Recall (and it is easy to see) that

$$f(m,n,k)=\sum_{r=0}^{\min(m,n,k)}\frac{(m+n+k-2r)!}{(m-r)!(n-r)!(k-r)!r!}.$$

Received date: 2003-05-26
Foundation item: the Natural Science Foundation of Education Department of Jiangsu Province (02KJB52005).

Jr. Stocks[4] obtained the following results:

$$Q(n) = n^2 + \sum_{r=0}^{n-2} \frac{(r+1)(3n-2r-2)! - (n-r-1)(n-r)^2(3n-2r-4)!}{(n-r)^2((n-r-1)!)^3 r!}, \tag{1}$$

and

$$Q'(n) = n+1+\sum_{r=0}^{n-1} \frac{(r+1)(3n-2r)!}{(n-r)(n-r-1)!(n-r)!(n-r+1)!r!} - \sum_{r=0}^{n-1} \frac{(3n-2r-2)!}{(n-r)((n-r-1)!)^3 r!}. \tag{2}$$

In this note, we give simpler formulas for $Q(n)$ and $Q'(n)$ by using a simpler method, and extend them to a general case.

2. Results and their proofs

Theorem 1 *Let $Q(n)$ and $Q'(n)$ be defined as above. Then we have*

(i)

$$Q(n) = n + \sum_{r=0}^{n-2} \frac{(3n-2r-2)!}{((n-r)!)^2 (n-r-1)! r!}, \tag{3}$$

(ii)

$$Q'(n) = 1 + \sum_{r=0}^{n-1} \frac{(3n-2r)!}{(n-r+1)((n-r)!)^3 r!}. \tag{4}$$

Proof (i) We use Andre's reflection principle[6−8].

Any path enumerated by $Q(n)$ must begin with step $(0,0,0) \to (1,0,0)$ and terminate with step $(n,n-1,n) \to (n,n,n)$. The total number of paths from $(1,0,0)$ to $(n,n-1,n)$ is $f(n-1,n-1,n)$.

Now we consider the paths from $(1,0,0)$ to $(n,n-1,n)$ which touch or cross the plane $y=x$. Any such a path must do so for the first time at a point (i,i,j). We construct a new path from $(0,1,0)$ to (i,i,j) which symmetric, with respect to the plane $y=x$, to the above path from $(1,0,0)$ to (i,i,j). As a result, we get a path from $(0,1,0)$ to $(n,n-1,n)$, the total number of which is $f(n,n-2,n)$. Obviously, the above reflection process is reversible. So we have

$$\begin{aligned} Q(n) &= f(n-1,n-1,n) - f(n,n-2,n) \\ &= \sum_{r=0}^{n-1} \frac{(3n-2-2r)!}{((n-1-r)!)^2 (n-r)! r!} - \sum_{r=0}^{n-2} \frac{(3n-2-2r)!}{((n-r)!)^2 (n-2-r)! r!} \\ &= n + \sum_{r=0}^{n-2} \frac{(3n-2-2r)!}{((n-r)!)^2 (n-r-1)! r!}. \end{aligned}$$

(ii) The number $Q'(n)$ is the same as the number of lattice paths from $(0,0,0)$ to $(n+1,n+1,n)$, which, except for the end points, contain only paths on the $(n,0,0)$ side of the plane $y=x$. Since $Q(n) = f(n-1,n-1,n) - f(n,n-2,n)$, we have

$$Q'(n) = f(n,n,n) - f(n+1,n-1,n)$$

$$=\sum_{r=0}^{n}\frac{(3n-2r)!}{((n-r)!)^3r!}-\sum_{r=0}^{n-1}\frac{(3n-2r)!}{(n-r)!(n-r-1)!(n-r+1)!r!}$$
$$=1+\sum_{r=0}^{n-1}\frac{(3n-2r)!}{(n-r+1)((n-r)!)^3r!}.$$

This completes the proof.

We now give a generalization of Theorem 1. Let

$Q(m,n,k)$= number of paths from $(0,0,0)$ to (m,n,k) (with $m>n$) which do not touch the plane $y=x$ except at $(0,0,0)$.

$Q'(m,n,k)$= number of paths from $(0,0,0)$ to (m,n,k) (with $m>n$) which may touch but not cross the plane.

Theorem 2 *Let $Q(m,n,k)$ and $Q'(m,n,k)$ be defined as above. Then we have*

(i)

$$Q(m,n,k)=\begin{cases}(m-n)\sum\limits_{r=0}^{k}\frac{(m+n+k-2r-1)!}{(m-r)!(n-r)!(k-r)!r!}, & \text{if } n>k,\\ \frac{(m-n+k-1)!}{(m-n-1)!(k-n)!n!}+(m-n)\sum\limits_{r=0}^{n-1}\frac{(m+n+k-2r-1)!}{(m-r)!(n-r)!(k-r)!r!}, & \text{if } n\le k;\end{cases}\tag{5}$$

(ii)

$$Q'(m,n,k)=\begin{cases}(m-n)\sum\limits_{r=0}^{k}\frac{(m+n+k-2r+1)!}{(m-r+1)!(n-r+1)!(k-r)!r!}, & \text{if } n+1>k,\\ \frac{(m-n+k-1)!}{(m-n-1)!(k-n-1)!(n+1)!}+(m-n)\sum\limits_{r=0}^{n}\frac{(m+n+k-2r+1)!}{(m-r+1)!(n-r+1)!(k-r)!r!}, & \text{if } n+1\le k.\end{cases}\tag{6}$$

Proof By using the Andre's reflection principle (see the proof of Theorem 1), we have

$$Q(m,n,k)=f(m-1,n,k)-f(m,n-1,k).$$

If $n>k$, then

$$Q(m,n,k)=f(m-1,n,k)-f(m,n-1,k)$$
$$=\sum_{r=0}^{k}\frac{(m+n+k-1-2r)!}{(m-1-r)!(n-r)!(k-r)!r!}-\sum_{r=0}^{k}\frac{(m+n+k-1-2r)!}{(m-r)!(n-1-r)!(k-r)!r!}$$
$$=(m-n)\sum_{r=0}^{k}\frac{(m+n+k-2r-1)!}{(m-r)!(n-r)!(k-r)!r!}.$$

If $n\le k$, then

$$Q(m,n,k)=f(m-1,n,k)-f(m,n-1,k)$$
$$=\sum_{r=0}^{n}\frac{(m+n+k-1-2r)!}{(m-1-r)!(n-r)!(k-r)!r!}-\sum_{r=0}^{n-1}\frac{(m+n+k-1-2r)!}{(m-r)!(n-1-r)!(k-r)!r!}$$
$$=\frac{(m-n+k-1)!}{(m-n-1)!(k-n)!n!}+(m-n)\sum_{r=0}^{n-1}\frac{(m+n+k-2r-1)!}{(m-r)!(n-r)!(k-r)!r!}.$$

(ii) Formula (6) follows from (5) and $Q'(m,n,k)=Q(m+1,n+1,k)$. The proof is complete.

Remark If we replace m,n,k by $n,n-1,n$ in (5), respectively, then we get (3); if we replace m,n,k by $n,n-1,n$ in (6) respectively, then we get (4).

Acknowledgement The author is indebted to Professor Zhi-Wei Sun for his helpful suggestions in the preparation of this paper.

References:

[1] CHOI S. *Counting lattice path in restricted planes* [J]. Discrete Math., 2000, **212**: 191–198.
[2] MOSER L, ZAYACHKOWSKI W. *Lattice paths with diagonal steps* [J]. Scripta Math., 1963, **26**: 223–229.
[3] PERGOLA E. *Two bijections for the area of Dyck paths* [J]. Discrete Math., 2001, **241**: 435–447.
[4] STOCKS D R Jr. *Lattice paths in E^3 with diagonal steps* [J]. Canad. Math. Bull., 1967, **10**: 653–658.
[5] WAGNER C G. *The Carlitz lattice path polynomials* [J]. Discrete Math., 2000, **222**: 291–298.
[6] COMTET L. *Advanced Combinatorics* [M]. Reidel, Dordrecht, 1974.
[7] KAPARTHI S, RAO H R. *Higher dimensional restricted lattice paths with diagonal steps* [J]. Discrete Appl. Math., 1991, **31**: 279–289.
[8] ZEILBERGER D. *Andre's reflection proof generalized to the many candidate ballot problem* [J]. Discrete Math., 1983, **44**: 325–326.

三维空间中带对角步格路的一个注记

卢青林
(徐州师范大学数学系，江苏 徐州 221116)

摘要: Jr. Stocks[4] 讨论了从 $(0,0,0)$ 到 (n,n,n) 的带对角步格路的计数问题. 本文给出了 [4] 中主要结果的简单公式，并将其推广到了一般情形.

关键词: 格路； 对角步； Andre 反射原理.