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Positive Solutions for Singular Boundary Value Problem of Fourth Order

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Abstract: Some existence results existence of the positive solutions for singular boundary value problems

$$\begin{cases} u^{(4)}(t) = p(t)f(u(t)), & t \in (0,1) \\ u(0) = u(1) = 0, \\ u'(0) = u'(1) = 0 \end{cases}$$

are given, where the function $p(t)$ may be singular at $t = 0, 1$.

Key words: singular boundary value problem; positive solution; variational method.
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1. Introduction

We consider the following problem

$$\begin{cases} u^{(4)}(t) = p(t)f(u(t)), & t \in (0,1) \\ u(0) = u(1) = 0, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $p \in C(0,1)$ and may be singular at $t = 0$ or $t = 1$. We are looking for positive classical solution for(1.1). A function u is called a classical solution of (1.1), if $u \in C[0,1] \bigcap C^4(0,1)$ and satisfies both the equation and the boundary value condition. The basic assumption on p is

(P) $p \geq 0, p \in C(0,1), \exists t \in (0,1), p(t) > 0$ and $\lim_{s\to 0} s^3 \int_s^{1-s} p(t)\mathrm{d}t = 0$.

For the nonlinear function f we assume:

(F) $f \geq 0$ and $f \in C[0,+\infty)$ and we denote

$$f_0^- = \underline{\lim}_{t\to 0} \frac{f(t)}{t}, \quad f_0^+ = \overline{\lim}_{t\to 0} \frac{f(t)}{t},$$

$$f_\infty^- = \underline{\lim}_{t\to +\infty} \frac{f(t)}{t}, \quad f_\infty^+ = \overline{\lim}_{t\to +\infty} \frac{f(t)}{t}.$$

In the following we will use the weighted L^2-space

$$L_p^2 = \{u | \int_0^1 p(s)u^2(s)\mathrm{d}s < +\infty\}$$

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equipped with the quasi-norm $|u|_p = \{\int_0^1 p(s)u^2(s)\mathrm{d}s\}^{\frac{1}{2}}$. We also need the Sobolev space H_0^2. The norm of H_0^2 is denoted by $\|\cdot\|$: $\|u\| = \{\int_0^1 |u''(s)|^2\mathrm{d}s\}^{\frac{1}{2}}$, and H_0^2 is the completion of $C_0^\infty(0,1)$ with respect to this norm.

Our main results are:

Theorem A *Let L_p^2 be the weighted L^2-space. Then*

(1) H_0^2 is continuously embedded into L_p^2 if and only if

$$s^3\int_s^{1-s} p(t)\mathrm{d}t \text{ is bounded as } s \to 0. \tag{1.2}$$

(2) H_0^2 is compactly embedded into L_p^2 if and only if

$$\lim_{s\to 0} s^3\int_s^{1-s} p(t)\mathrm{d}t = 0. \tag{1.3}$$

Let

$$\lambda = \inf_{u\in H_0^2} \frac{\int_0^1 |u''(t)|^2\mathrm{d}t}{\int_0^1 p(t)u^2(t)\mathrm{d}t}. \tag{1.4}$$

As a consequence of Theorem A, $\lambda > 0$ is achieved if (1.3) holds.

Theorem B *Assume that (P) and (F) hold . Assume moreover that either*

$$f_0^+ < \lambda < f_\infty^- \le f_\infty^+ \le +\infty \tag{1.5}$$

or

$$f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty. \tag{1.6}$$

Then Problem (1.1) has a positive classical solution.

In the following the variational method is used. Define $f(t) = 0$, as $t \le 0$. Let F be the primitive function of f, $F(t) = \int_0^t f(s)\mathrm{d}s$. Define a functional I on H_0^2

$$I(u) = \frac{1}{2}\int_0^1 |u''(t)|^2\mathrm{d}t - \int_0^1 p(t)F(u(t))\mathrm{d}t.$$

I is well-defined on H_0^2 and C^1-continuous. A sequence $\{u_n\} \subset H_0^2$ such that $I(u_n) \to c$ as $n \to \infty$, where c is a constant, and $I'(u_n) \to 0$ in H_0^{-2} will be called a Palais-Smale sequence for the functional I. We say that a sequence $\{u_n\} \subset H_0^2$ satisfies the Palais-Smale condition (P.S condition for short) if it is a Palais-Smale sequence and is relatively compact. Any critical point u of the functional I is a weak solution of equation (1.1)

$$\int_0^1 u''(t)\varphi''(t)\mathrm{d}t - \int_0^1 p(t)f(u(t))\varphi(t)\mathrm{d}t = 0\ ,\ \forall\varphi \in H_0^2. \tag{1.7}$$

It is easy to prove that such a weak solution is a positive classical solution of (1.1).

This paper is organized as follows . In Section 2, we prove the embedding Theorem A. In Section 3, we prove the existence Theorem B under the assumption $f_\infty^+ < +\infty$. In Section 4, we deal with the case $f_\infty^+ = +\infty$.

2. Proof of Theorem A

In this section we prove the embedding Theorem A. To simplify the presentation, we assume that p has only a unique singular point $t=0$, $p \in C(0,1]$. It is easy to see

$$|u'(t)|^2 = (\int_0^t u''(s)\mathrm{d}s)^2 \le t\int_0^t |u''(s)|^2\mathrm{d}s \le t\|u\|^2, \tag{2.1}$$

$$u^2(t) = (\int_0^t u'(s)\mathrm{d}s)^2 \le t\int_0^t |u'(s)|^2\mathrm{d}s \le t\int_0^t s\|u\|^2\mathrm{d}s \le \frac{1}{2}t^3\|u\|^2, \tag{2.2}$$

$$u(t) = t\int_0^1 u'(\theta t)\mathrm{d}\theta. \tag{2.3}$$

By integration by parts, we have

$$\int_0^1 p(t)u^2(t)\mathrm{d}t = [-u^2(t)\int_t^1 p(t)\mathrm{d}s]_0^1 + 2\int_0^1 u'(t)u(t)(\int_t^1 p(s)\mathrm{d}s)\mathrm{d}t. \tag{2.4}$$

From (2.1), (2.2) and $t^3\int_t^1 p(s)\mathrm{d}s < +\infty$, we have the first term of (2.4) $I_1 \le c_1\|u\|^2$. On the second term of the (2.4), we have

$$\begin{aligned}
I_2 &\le c_2\int_0^1 |\frac{u(t)}{t^2}||\frac{u'(t)}{t}|\mathrm{d}t \\
&\le c_2\int_0^1 |\frac{\int_0^1 u'(\theta t)\mathrm{d}\theta}{t}||\frac{u'(t)}{t}|\mathrm{d}t \\
&\le c_2\{\int_0^1 |\frac{\int_0^1 u'(\theta t)\mathrm{d}\theta}{t}|^2\mathrm{d}t\}^{\frac{1}{2}}\{\int_0^1 |\frac{u'(t)}{t}|^2\mathrm{d}t\}^{\frac{1}{2}} \\
&\le c_2\int_0^1\{\int_0^1 |u''(\theta t)\theta|^2\mathrm{d}t\}^{\frac{1}{2}}\mathrm{d}\theta\{\int_0^1 |u''(t)|^2\mathrm{d}t\}^{\frac{1}{2}} \\
&\le c_2\int_0^1\{\int_0^1 |u''(\theta t)|^2\theta\mathrm{d}t\}^{\frac{1}{2}}\mathrm{d}\theta\{\int_0^1 |u''(t)|^2\mathrm{d}t\}^{\frac{1}{2}} \\
&\le c_2\|u\|^2.
\end{aligned}$$

Here we have used the Hardy inequality

$$\int_0^{+\infty} |\frac{u(t)}{t}|^p\mathrm{d}t \le \int_0^{+\infty} |u'(t)|^p\mathrm{d}t\,, \ \text{provided } u(0)=0.$$

Hence we have the continuous embedding $H_0^2 \hookrightarrow L_p^2$, provided the quantity $t^3\int_t^1 p(s)\mathrm{d}s < +\infty$ keeps bounded. Now suppose that $t^3\int_t^1 p(s)\mathrm{d}s \to 0$, as $t \to 0$. We are going to show that this embedding is compact. Let $\{u_n\}$ be a bounded subset of H_0^2. We can assume that $\{u_n\}$ uniformly converges to a function u in $C^1[0,1]$. We have

$$\begin{aligned}
&\int_0^1 p(t)|u_n(t)-u(t)|^2\mathrm{d}t \\
&= \lim_{t\to 0}|u_n-u|^2\int_t^1 p(s)\mathrm{d}s + 2\int_0^1 p(t)(u_n(t)-u(t))(u_n'(t)-u'(t))(\int_t^1 p(s)\mathrm{d}s)\mathrm{d}t.
\end{aligned}\tag{2.5}$$

By (2.2) we have that the first term of the right hand of (2.5) is 0. On the other hand, by $\lim_{t\to 0} t^3\int_t^1 p(s)\mathrm{d}s = 0, \forall\varepsilon > 0$, $\exists\delta > 0$, as $0 < t < \delta$, $0 < t^3\int_t^1 p(s)\mathrm{d}s < \varepsilon$. So the second term of the right hand of (2.5) is

$$\begin{aligned}&2\left|\int_0^1 (u_n(t)-u(t))(u_n'(t)-u'(t))\left(\int_t^1 p(s)\mathrm{d}s\right)\mathrm{d}t\right|\\ &\le 2\left(\int_0^\delta+\int_\delta^1\right)|u_n'(t)-u'(t)||u_n(t)-u(t)|\left(\int_t^1 p(s)\mathrm{d}s\right)\mathrm{d}t. \end{aligned}\tag{2.6}$$

It is easy to see that

$$\begin{aligned}&\int_0^\delta |u_n'(t)-u'(t)||u_n(t)-u(t)|\left(\int_t^1 p(s)\mathrm{d}s\right)\mathrm{d}t\\ &=\int_0^\delta \left|\frac{u_n'(t)-u'(t)}{t}\right|\left|\frac{u_n(t)-u(t)}{t^2}\right|\left(t^3\int_t^1 p(s)\mathrm{d}s\right)\mathrm{d}t\\ &\le\varepsilon\int_0^\delta \left|\frac{u_n'(t)-u'(t)}{t}\right|\left|\frac{u_n(t)-u(t)}{t^2}\right|\mathrm{d}t\\ &\le\varepsilon\int_0^\delta \left|\frac{u_n'(t)-u'(t)}{t}\right|\left|\frac{\int_0^1(u_n'(\theta t)-u'(\theta t))\mathrm{d}\theta}{t}\right|\mathrm{d}t\\ &\le\varepsilon\left\{\int_0^\delta \left|\frac{u_n'(t)-u'(t)}{t}\right|^2\mathrm{d}t\right\}^{\frac12}\left\{\int_0^\delta\left|\frac{\int_0^1(u_n'(\theta t)-u'(\theta t))\mathrm{d}\theta}{t}\right|^2\mathrm{d}t\right\}^{\frac12}\\ &\le\varepsilon\left\{\int_0^\delta |u_n''(t)-u''(t)|^2\mathrm{d}t\right\}^{\frac12}\left\{\int_0^\delta\left|\int_0^1(u_n''(\theta t)-u''(\theta t))\theta\mathrm{d}\theta\right|^2\mathrm{d}t\right\}^{\frac12}\\ &\le\varepsilon\left\{\int_0^\delta |u_n''(t)-u''(t)|^2\mathrm{d}t\right\}^{\frac12}\int_0^\delta\left\{\int_0^1|(u_n''(\theta t)-u''(\theta t)|^2\theta\mathrm{d}t\right\}^{\frac12}\mathrm{d}\theta\\ &=\varepsilon\|u_n-u\|^2.\end{aligned}\tag{2.7}$$

Since $\int_t^1 p(s)\mathrm{d}s$ is bounded on $[\delta,1]$, say $\int_t^1 p(s)\mathrm{d}s < C_\varepsilon$, so we have

$$\begin{aligned}&\int_\delta^1 |u_n'(t)-u'(t)||u_n(t)-u(t)|\left(\int_t^1 p(s)\mathrm{d}s\right)\mathrm{d}t \le C_\varepsilon\int_\delta^1 |u_n'(t)-u'(t)||u_n(t)-u(t)|\mathrm{d}t\\ &\le C_\varepsilon\|u_n-u\|_{c[0,1]}\|u_n'-u'\|_{c[0,1]}.\end{aligned}\tag{2.8}$$

From (2.7) and (2.8) we have

$$\int_0^1 p(t)|u_n(t)-u(t)|^2\mathrm{d}t \le \mu(\varepsilon)+C_\varepsilon\|u_n-u\|_{c[0,1]}\|u_n'-u'\|_{c[0,1]}.\tag{2.9}$$

$\mu(\varepsilon)$ denotes a small quantity which tends to zero as $\varepsilon\to 0$ and C_ε denotes constants dependent on ε . From (2.9), we have $\int_0^1 p(t)|u_n(t)-u(t)|^2\mathrm{d}t\to 0$, as $n\to\infty$. This completes the sufficient part of Theorem A.

Now suppose that H_0^2 is continuously embedded into L_p^2. For $0<\varepsilon<\frac12$, define a function

u_ε by

$$u_\varepsilon(t)=\begin{cases}\frac{2}{\sqrt{\varepsilon}}t^2, & 0\le t\le \frac{\varepsilon}{2},\\ -\frac{2}{\sqrt{\varepsilon}}(t-\varepsilon)^2+\varepsilon^{\frac{3}{2}}, & \frac{\varepsilon}{2}\le t\le\varepsilon,\\ \varepsilon^{\frac{3}{2}}, & \varepsilon\le t\le 1-\varepsilon,\\ -\frac{2}{\sqrt{\varepsilon}}(t-1+\varepsilon)^2+\varepsilon^{\frac{3}{2}}, & 1-\varepsilon\le t\le 1-\frac{\varepsilon}{2},\\ \frac{2}{\sqrt{\varepsilon}}(t-1)^2, & 1-\frac{\varepsilon}{2}\le t\le 1.\end{cases}\tag{2.10}$$

Then $u_\varepsilon(t)\in H_0^2$ and $\|u_\varepsilon\|^2=32$. Suppose that H_0^2 is continuously embedded into L_p^2, then we get

$$c\ge\int_0^1 p(t)u_\varepsilon^2(t)\mathrm{d}t\ge\int_\varepsilon^{1-\varepsilon}p(t)u_\varepsilon^2(t)\mathrm{d}t=\varepsilon^3\int_\varepsilon^{1-\varepsilon}p(t)\mathrm{d}t.\tag{2.11}$$

Suppose now that H_0^2 is compactly embedded into L_p^2. Since the function u_ε defined by (2.10) uniformly converges to zero , hence by compactness u_ε converges to zero in L_p^2. And from (2.11) we have

$$\varepsilon^3\int_\varepsilon^{1-\varepsilon}p(t)\mathrm{d}t\le\int_0^1 p(t)u_\varepsilon^2(t)\mathrm{d}t\to 0\ ,\ \text{as } \varepsilon\to 0.$$

The proof of Theorem A is completed.

Let $G(t,s)$ be the Green function to (1.1) denoted by

$$G(t,s)=\begin{cases}\frac{1}{6}(1-t)^2s^2[(t-s)+2(1-s)t]\ ,\ 0\le s\le t\le 1,\\ \frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s]\ ,\ 0\le t\le s\le 1.\end{cases}$$

Lemma 2.1 *Let*

$$\lambda=\inf_{u\in H_0^2}\frac{\int_0^1|u''(t)|^2\mathrm{d}t}{\int_0^1 p(t)u^2(t)\mathrm{d}t}.$$

There exists a function $\varphi\in H_0^2$ such that $\varphi>0$, $\forall t\in(0,1)$, $\frac{\int_0^1|\varphi''(t)|^2\mathrm{d}t}{\int_0^1 p(t)\varphi^2(t)\mathrm{d}t}=\lambda$, and φ satisfies

$$\begin{cases}\varphi^{(4)}(t)=\lambda p(t)\varphi(t),\quad t\in(0,1),\\ \varphi(0)=\varphi(1)=0,\\ \varphi'(0)=\varphi'(1)=0.\end{cases}\tag{2.12}$$

Proof The existence of a minimizer φ is a consequence of Theorem A (2) , and φ satisfies the weak form of (2.12)

$$\int_0^1\varphi''(t)v''t)\mathrm{d}t=\lambda\int_0^1 p(t)\varphi(t)v(t)\mathrm{d}t,\quad\forall v\in H_0^2.\tag{2.13}$$

By the regularity theory (2.12) follows from (2.13). It is obviously φ is also the solution of the integral equation

$$\varphi(t)=\lambda\int_0^1 G(t,s)p(s)\varphi(s)\mathrm{d}s.$$

Let w be the solution of the following equation

$$\begin{cases}w^{(4)}(t)=\lambda p(t)|\varphi(t)|,\quad t\in(0,1),\\ w(0)=w(1)=0,\\ w'(0)=w'(1)=0.\end{cases}\tag{2.14}$$

Then we have

$$w(t)=\lambda\int_0^1 G(t,s)p(s)|\varphi(s)|\mathrm{d}s\ge|\lambda\int_0^1 G(t,s)p(s)\varphi(s)\mathrm{d}s|=|\varphi(t)|.$$

On the other hand, let $\mu=\inf\limits_{u\in H_0^2}\frac{\int_0^1|u''(t)|^2\mathrm{d}t}{\int_0^1 p(t)(u^+)^2(t)\mathrm{d}t}$. By Theorem A, there exists $u\in H_0^2$, which satisfies

$$\int_0^1 u''(t)v''(t)\mathrm{d}t=\mu\int_0^1 p(t)u^+(t)v(t)\mathrm{d}t\ ,\ \forall v\in H_0^2. \tag{2.15}$$

It is obvious that $\int_0^1 p(t)(u^+)^2(t)\mathrm{d}t\neq 0$ and $\mu\ge\lambda$. Similarly, we have

$$\begin{aligned}u(t)&=\int_0^1 G(t,s)p(s)u^+(s)\mathrm{d}s\\&\ge\frac{1}{3}(1-t)^3t\int_0^t s^2p(s)u^+(s)\mathrm{d}s+\frac{1}{3}t^2\int_t^1 s(1-s)^3p(s)u^+(s)\mathrm{d}s>0.\end{aligned} \tag{2.16}$$

We can prove $\mu=\lambda$. In fact we have

$$\begin{aligned}\mu\int_0^1 p(t)u(t)w(t)\mathrm{d}t&\le\mu\int_0^1 p(t)u^+(t)w(t)\mathrm{d}t=\int_0^1 u''(t)w''(t)\mathrm{d}t\\&=\lambda\int_0^1 p(t)u(t)|\varphi(t)|\mathrm{d}t\le\lambda\int_0^1 p(t)u(t)w(t)\mathrm{d}t.\end{aligned}$$

So we have $\mu\le\lambda$, and $\mu=\lambda$. This means that the existence λ is achieved at a nonnegative function φ. In turn (2.16) implies that $\varphi>0,\forall t\in(0,1)$.

3. Proof of Theorem B for the case $f_\infty^+<+\infty$

In this section, we will prove Theorem B in the case $f_\infty^+<+\infty$ by variational method, especially the Mountain Pass Lemma. Firstly, we verify the P.S condition.

Lemma 3.1 *Suppose (P), (F) hold and $f_\infty^+<+\infty$, $f_0^+<+\infty$. I is well-defined on H_0^2 and C^1-continuous. The Fréchet derivative of I has a form*

$$<I'(u),\varphi>=\int_0^1 u''(t)\varphi''(t)\mathrm{d}t-\int_0^1 p(t)f(u(t))\varphi(t)\mathrm{d}t\ ,\ \forall\varphi\in H_0^2. \tag{3.1}$$

Proof Since $f_\infty^+<+\infty$ and $f_0^+<+\infty$, we have a constant c such that $0\le f(t)\le c|t|$, $0\le F(t)\le ct^2$. By theorem A , I is well-defined on H_0^2. For $\varphi\in H_0^2$, we have

$$\begin{aligned}&|I(u+\varphi)-I(u)-\int_0^1 u''\varphi''dt+\int_0^1 p(t)f(u(t))\varphi(t)\mathrm{d}t|\\&\le\frac{1}{2}\|\varphi\|^2+|\int_0^1 p(t)f(u)\varphi\mathrm{d}t-\int_0^1 p(t)(F(u+\varphi)-F(u))\mathrm{d}t|\\&\le\frac{1}{2}\|\varphi\|^2+\int_0^1 p(t)|f(u+\theta\varphi)-f(u)||\varphi|\mathrm{d}t\\&\le\frac{1}{2}\|\varphi\|^2+|f(u+\theta\varphi)-f(u)|_p|\varphi|_p\\&\le\frac{1}{2}\|\varphi\|^2+c|f(u+\theta\varphi)-f(u)|_p\|\varphi\|,\end{aligned} \tag{3.2}$$

where $0<\theta(t)<1$. Since $|f(u+\theta\varphi)|\leq c(|u|+|\varphi|)$ and $\int_0^1 p(t)(|u|+|\varphi|)|\varphi|\mathrm{d}t<+\infty$, by the Lebesque dominant theorem $|f(u+\theta\varphi)-f(u)|_p$ tends to zero as $\|\varphi\|\to 0$. So Lemma 3.1 is true.

Lemma 3.2 *Suppose (P) , (F) hold and either $f_0^+<\lambda<f_\infty^-\leq f_\infty^+<+\infty$ or $f_\infty^+<\lambda<f_0^-\leq f_0^+<+\infty$, then I satisfies the P.S condition.*

Proof Let $\{u_n\}$ be a P.S sequence, then

$$<I'(u_n),\varphi>=\int_0^1 u_n''\varphi''\mathrm{d}t-\int_0^1 p(t)f(u_n)\varphi\mathrm{d}t=o(\|\varphi\|)\ ,\ \forall\varphi\in H_0^2. \tag{3.3}$$

If $|u_n^+|_p$ is bounded, where $u^+=\max(u,0)$, then taking $\varphi=u_n$ in (3.3) and noticing that $|f(u_n)|\leq cu_n^+$, we have a bound of $\|u_n\|$. Let $u_n\rightharpoonup u$ in H_0^2. By the embedding theorem A, we can assume that $u_n\to u$ in L_p^2, hence $f(u_n)\to f(u)$ in L_p^2. It follows from (3.3) that $u_n\to u$ in H_0^2.

Now we prove that any P.S sequence $\{u_n\}$ is bound in L_p^2 by a contradiction argument. Otherwise suppose $|u_n^+|_p\to+\infty$. Set $v_n=\frac{u_n}{|u_n^+|_p}$, $|v_n^+|_p=1$. Taking $\varphi=u_n$ in (3.3), we have that $\|v_n\|$ is bounded. Assume that $v_n\rightharpoonup v$ in H_0^2, $v_n\to v$ in $C[0,1]$ and in L_p^2, $|v^+|_p=1$. From (3.3) we have

$$\int_0^1 v_n''\varphi''\mathrm{d}t=\int_0^1 p(t)\frac{f(u_n)}{|u_n^+|_p}\varphi\mathrm{d}t+\frac{o(\|\varphi\|)}{|u_n^+|_p}. \tag{3.4}$$

We firstly consider the case $f_0^+<\lambda<f_\infty^-\leq f_\infty^+<+\infty$. Let $\varepsilon>0$ and $f_\infty^--\varepsilon>\lambda$, and choose a constant $M>0$ such that $f(t)\geq(f_\infty^--\varepsilon)t$, $\forall t>M$. Let $\varphi\in H_0^2$ and $\varphi\geq 0$. From (3.4), we have

$$\begin{aligned}\int_0^1 v_n''\varphi''\mathrm{d}t&=\int_0^1 p(t)\frac{f(|u_n^+|_pv_n)}{|u_n^+|_p}\varphi\mathrm{d}t+o(1)\\&=\int_{|u_n^+|_pv_n<M}p(t)\frac{f(|u_n^+|_pv_n)}{|u_n^+|_p}\varphi\mathrm{d}t+\int_{|u_n^+|_pv_n\geq M}p(t)\frac{f(|u_n^+|_pv_n)}{|u_n^+|_p}\varphi\mathrm{d}t+o(1)\\&\geq\int_{|u_n^+|_pv_n<M}p(t)\frac{f(|u_n^+|_pv_n)}{|u_n^+|_p}\varphi\mathrm{d}t+(f_\infty^--\varepsilon)\int_{|u_n^+|_pv_n\geq M}p(t)v_n^+\varphi\mathrm{d}t+o(1)\\&\geq(f_\infty^--\varepsilon)\int_0^1 p(t)v_n^+\varphi\mathrm{d}t-c\int_{0\leq|u_n^+|_pv_n\leq M}p(t)v_n^+\varphi\mathrm{d}t+o(1).\end{aligned} \tag{3.5}$$

Let $n\to\infty$, we have

$$\int_0^1 v''\varphi''\mathrm{d}t\geq(f_\infty^--\varepsilon)\int_0^1 p(t)v^+\varphi\mathrm{d}t\ ,\ \forall\varphi\in H_0^2\ ,\ \varphi\geq 0. \tag{3.6}$$

In particular, let φ be the minimizer in Lemma 2.1, then we have

$$\lambda\int_0^1 p(t)v^+\varphi\mathrm{d}t\geq\lambda\int_0^1 p(t)v\varphi\mathrm{d}t=\int_0^1 v''\varphi''\mathrm{d}t\geq(f_\infty^--\varepsilon)\int_0^1 p(t)v^+\varphi\mathrm{d}t, \tag{3.7}$$

which implies that $pv^+\equiv 0$. Hence $|v^+|_p=0$, a contradiction.

Suppose now that $f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty$. Let $\varepsilon > 0$ and $f_\infty^+ + \varepsilon < \lambda$. Similar to (3.5), we have

$$\int_0^1 v_n''\varphi''\mathrm{d}t \le (f_\infty^+ + \varepsilon)\int_0^1 p(t)v_n^+\varphi\mathrm{d}t + c\int_{0\le|u_n^+|_p v_n\le M} p(t)v_n^+\varphi\mathrm{d}t + o(1),$$

and as $n\to\infty$ we have

$$\int_0^1 v''\varphi''\mathrm{d}t \le (f_\infty^+ + \varepsilon)\int_0^1 p(t)v^+\varphi\mathrm{d}t\ ,\ \forall\varphi\in H_0^2\ ,\ \varphi\ge 0. \tag{3.8}$$

Taking $\varphi = v^+$ in (3.8) , by the definition of λ we have

$$\lambda\int_0^1 p(t)(v^+)^2\mathrm{d}t \le \int_0^1 |(v^+)''|^2\mathrm{d}t \le (f_\infty^+ + \varepsilon)\int_0^1 p(t)(v^+)^2\mathrm{d}t.$$

Since $\int_0^1 p(t)(v^+)^2\mathrm{d}t = 1$, we arrive at a contradiction. Hence I satisfies the P.S condition.

Proof of Theorem B The case $f_\infty^+ < \lambda < f_0^- \le f_0^+ < +\infty$. We prove that in this case the functional I is coercive, that is $I(u)\to\infty$ as $\|u\|\to\infty$. We use a contradiction argument. Suppose that there is a sequence $\{u_n\}\subset H_0^2$ such that $\|u_n\|\to\infty$, but $I(u_n)\le c < +\infty$. Set $v_n = \frac{u_n}{|u_n^+|_p}$, then we have $|v_n^+|_p = 1$ and

$$c \ge I(u_n) = \frac{1}{2}\int_0^1 |u_n''|^2\mathrm{d}t - \int_0^1 p(t)F(u_n(t))\mathrm{d}t.$$

Dividing by $|u_n^+|_p^2$, this gives

$$\int_0^1 p(t)\frac{F(u_n(t))}{|u_n^+|_p^2}\mathrm{d}t \ge o(1) + \frac{1}{2}\lambda.$$

Let $\varepsilon > 0$ and $f_\infty^+ + \varepsilon < \lambda$, and choose a constant $M > 0$ such that $F(t) \le \frac{1}{2}(f_\infty^+ + \varepsilon)t^2$, $\forall t > M$. Hence we have

$$\int_0^1 p(t)\frac{F(u_n(t))}{|u_n^+|_p^2}\mathrm{d}t \le \frac{1}{2}(f_\infty^+ + \varepsilon)\int_0^1 p(t)(v_n^+)^2\mathrm{d}t + c\int_{0\le|u_n^+|_p v_n\le M} p(t)(v_n^+)^2\mathrm{d}t.$$

Assuming $v_n\to v$ in L_p^2, we have $\lambda\le f_\infty^+ + \varepsilon$, a contradiction.

I is bounded from below and satisfies the P.S condition, hence I has a minimizer u. We need only to show that the trivial solution $u\equiv 0$ is not a local minimizer, then u_0 is a nontrivial classical positive solution. Let φ be the eigenfunction in Lemma 2.1, $\int_0^1|\varphi''|^2\mathrm{d}t = \lambda\int_0^1 p\varphi^2\mathrm{d}t$. For $\sigma > 0$ very small, we have

$$I(\sigma\varphi) = \frac{1}{2}\sigma^2\int_0^1 |\varphi''|^2\mathrm{d}t - \int_0^1 pF(\sigma\varphi)\mathrm{d}t.$$

Let $\varepsilon > 0$ and $f_0^- - \varepsilon > \lambda$, and choose a constant $\sigma > 0$ such that $F(t)\ge\frac{1}{2}(f_0^- - \varepsilon)t^2$, $\forall 0 < t < \sigma$.

Therefore, we have

$$\begin{aligned} I(\sigma\varphi) &\le \frac{1}{2}\sigma^2\int_0^1 |\varphi''|^2 \mathrm{d}t - \frac{1}{2}(f_0^- - \varepsilon)\sigma^2\int_0^1 p(t)\varphi^2 \mathrm{d}t \\ &= \frac{1}{2}\sigma^2\|\varphi\|^2 - \frac{1}{2\lambda}(f_0^- - \varepsilon)\sigma^2\|\varphi\|^2 \\ &= \frac{1}{2}\sigma^2\|\varphi\|^2(1 - \frac{f_0^- - \varepsilon}{\lambda}) < 0. \end{aligned}$$

Proof of Theorem B $f_0^+ < \lambda < f_\infty^- \le f_\infty^+ < +\infty$. In this case we use the mountain pass lemma. We need to verify the following:

(a) There are constants α, $\rho > 0$ such that $I(u) \ge \alpha$, $\forall u$, $\|u\| = \rho$.

(b) There is an element e such that $I(e) \le 0$ and $\|e\| > \rho$.

Take $\varepsilon > 0$, $f_0^+ + \varepsilon < \lambda$. For $\rho \ll 1$, $\|u\| = \rho$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^1 (u'')^2 \mathrm{d}t - \int_0^1 p(t)F(u)\mathrm{d}t \\ &\ge \frac{1}{2}\int_0^1 (u'')^2 dt - \frac{1}{2}(f_0^+ + \varepsilon)\int_0^1 p(t)u^2 \mathrm{d}t \\ &\ge \frac{1}{2}(1 - \frac{f_0^+ + \varepsilon}{\lambda})\int_0^1 (u'')^2 \mathrm{d}t \\ &= \frac{1}{2}(1 - \frac{f_0^+ + \varepsilon}{\lambda})\rho^2 = \alpha > 0. \end{aligned}$$

On the other hand, let φ be the eigenfunction in Lemma 2.1. Let $\varepsilon > 0$ and $f_\infty^- - \varepsilon > \lambda$, $F(t) \ge \frac{1}{2}(f_\infty^- - \varepsilon)t^2 - C$, $\forall t > 0$, then we have

$$\begin{aligned} I(T\varphi) &= \frac{1}{2}T^2\int_0^1 (\varphi'')^2 \mathrm{d}t - \int_0^1 p(t)F(T\varphi)\mathrm{d}t \\ &\le \frac{1}{2}T^2\int_0^1 (\varphi'')^2 \mathrm{d}t - \int_\delta^{1-\delta} p(t)F(T\varphi)\mathrm{d}t \\ &\le \frac{1}{2}T^2\|\varphi\|^2 - \frac{1}{2}T^2(f_\infty^- - \varepsilon)\int_\delta^{1-\delta} p(t)\varphi^2 \mathrm{d}t + C\int_\delta^{1-\delta} p(t)\mathrm{d}t. \end{aligned}$$

We can choose δ such that $\|\varphi\|^2 - (f_\infty^- - \varepsilon)\int_\delta^{1-\delta} p(t)\varphi^2 \mathrm{d}t < 0$, and T large enogh such that $\frac{1}{2}T^2[\|\varphi\|^p - (f_\infty^- - \varepsilon)\int_\delta^{1-\delta} p(t)\varphi^p \mathrm{d}t] + C\int_\delta^{1-\delta} p(t)\mathrm{d}t < 0$.

Now by the mountain pass lemma , we define

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma | \gamma \in C([0,1], H_0^2), \gamma(0) = \theta, \gamma(1) = T\varphi\},$$

$c \ge \alpha$ is a critical value of I, and I has a critical point u with $I(u) = c$. u is a classical positive solution of our problem (1.1).

4. Proof of Theorem B for the case $f_\infty^+ = +\infty$

In this section we deal with the case $f_0^+ < \lambda < f_\infty^- \le f_\infty^+ = \infty$. For $\delta > 0$. Define

$$\lambda_\delta = \inf_{u \in H_0^2} \frac{\int_0^1 |u''(t)|^2 \mathrm{d}t}{\int_\delta^{1-\delta} p(t)u^2(t)\mathrm{d}t},$$

then $\lambda_\delta \to \lambda$ as $\delta \to 0$. There is a function $\varphi_\delta \in H_0^2$ such that $\varphi_\delta(t) > 0, t \in (0,1), \lambda_\delta = \frac{\int_0^1 |\varphi_\delta''(t)|^2 \mathrm{d}t}{\int_\delta^{1-\delta} p(t)\varphi_\delta^2(t)\mathrm{d}t}$ and $\int_0^1 \varphi_\delta''(t)v''(t)\mathrm{d}t = \lambda_\delta \int_\delta^{1-\delta} p(t)\varphi_\delta(t)v(t)\mathrm{d}t$, $v \in H_0^2$. The existence of such λ_δ and φ_δ is obvious. To show that $\lambda_\delta \to \lambda$, let φ be the eigenfunction in Lemma 2.2. We have

$$\lambda \le \lambda_\delta \le \frac{\int_0^1 |\varphi''(t)|^2 \mathrm{d}t}{\int_\delta^{1-\delta} p(t)\varphi^2(t)\mathrm{d}t} \to \frac{\int_0^1 |\varphi''(t)|^2 \mathrm{d}t}{\int_0^1 p(t)\varphi^2(t)\mathrm{d}t} = \lambda$$

as $\delta \to 0$. Choose δ so small that $\lambda_\delta < f_\infty^-$ and $\exists t \in [\delta, 1-\delta]$, $p(t) \ne 0$. Choose M, Λ such that $\lambda_\delta < \Lambda < f_\infty^-$. $f(t) \ge \Lambda t$ when $t \ge \delta^2(1-\delta)M$. We define a function

$$g(t) = \begin{cases} \Lambda(t-M) + f(M), & t \ge M, \\ f(t), & t \le M. \end{cases}$$

It is easy to see $g(t) \ge \Lambda t$ as $t \ge \delta^2(1-\delta)M$. For this truncated function, by the result in Section 3 we have a solution u satisfing

$$\begin{cases} u^{(4)}(t) = p(t)g(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

or the weak form

$$\int_0^1 u''(t)\varphi''(t)\mathrm{d}t = \int_0^1 p(t)g(u(t))\varphi(t)\mathrm{d}t\,,\ \forall \varphi \in H_0^2.$$

We prove that the solution u of (4.1) satisfies $0 \le u \le M$ and hence is a solution of (1.1). Suppose that this is not true, then $\|u\|_{C[0,1]} > M$. By the following Lemma 4.1, we have that $u(t) \ge \delta^2 M$ for $t \in [\delta, 1-\delta]$ and $g(u(t)) \ge \Lambda u(t)$, for $t \in [\delta, 1-\delta]$. Now for every $v \in H_0^2$, $v \ge 0$ we have

$$\int_0^1 u''(t)v''(t)\mathrm{d}t \ge \int_\delta^{1-\delta} p(t)g(u(t))v(t)\mathrm{d}t \ge \Lambda \int_\delta^{1-\delta} p(t)u(t)v(t)\mathrm{d}t. \tag{4.2}$$

In particular, take $v = \varphi_\delta$, the minimizer for the eigenvalue λ_δ, then

$$\lambda_\delta \int_\delta^{1-\delta} p(t)u(t)\varphi_\delta(t)\mathrm{d}t = \int_0^1 u''(t)\varphi_\delta''(t)\mathrm{d}t \ge \Lambda \int_\delta^{1-\delta} p(t)u(t)\varphi_\delta(t)\mathrm{d}t,$$

which implies that $\int_\delta^{1-\delta} p(t)u(t)\varphi_\delta(t)\mathrm{d}t = 0$ and $p(t)u(t)\varphi_\delta(t) \equiv 0$ in $[\delta, 1-\delta]$. A contradiction. We have proved that $\max u \le M$ and u is the desired solution for our original problem.

Lemma 4.1 *Suppose $u \in C^2[0,1] \bigcap C^4(0,1)$ and $u^{(4)}(t) \ge 0$. Then $u(t) \ge \delta^2 \|u\|_{C[0,1]}$ for $t \in [\delta, 1-\delta]$.*

Proof Let $u(a) = M = \|u\|_{C[0,1]}$, then $u'(a) = 0$. For simplicity we assume that $\delta < a < 1 - \delta$. The other case could be treated in the same way.

Let φ be the solution of the equation

$$\begin{cases} \varphi^{(4)}(t) = 0, \quad t \in (0,1), \\ \varphi(0) = \varphi'(0) = 0, \varphi(a) = M, \varphi'(a) = 0. \end{cases}$$

The explicit form of φ is

$$\varphi(t) = \frac{M}{a^3} t^2 (3a - 2t).$$

Let $w(t) = u(t) - \varphi(t)$, then

$$\begin{cases} w^{(4)}(t) \geq 0, \quad t \in (0,1), \\ w(0) = w'(0) = 0, w(a) = w'(a) = 0. \end{cases}$$

By the Maximum Principle (or the form for the Green function) we have $w \geq 0$ in $[0, a]$. In particular

$$u(t) \geq \varphi(t) \geq \delta^2 M\ , \quad \text{for} \quad \delta \leq t \leq a.$$

Similary, $u(t) \geq \delta^2 M$, for $a \leq t \leq 1 - \delta$.

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一类四阶奇异边值问题的正解

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摘要: 本文讨论了如下四阶奇异边值问题正解的存在性

$$\begin{cases} u^{(4)} = p(t) f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u'(0) = u'(1) = 0, \end{cases}$$

其中 p 可能在 $t = 0, 1$ 都有奇点.

关键词: 奇异边值问题; 正解; 变分法.