第25卷第4期 数 学 研 究 与 评 论 Vol.25, No.4
2005年11月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Nov., 2005

Article ID: 1000-341X(2005)04-0583-06 **Document code**: A

A New Theorem of Existence of Solutions to Nonlinear Three-Point Boundary Value Problem

SUN Jian-ping, ZHAO Ya-hong

(Dept. of Appl. Math., Lanzhou University of Technology, Gansu 730050, China)
(E-mail: jpsun@lut.cn)

Abstract: A new theorem of existence of solutions for the nonlinear three-point boundary value problem

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+h(t)f(u)=0, t\in(0,1) \\ u(0)=0, u(1)=\alpha u(\eta) \end{cases}$$

is obtained by using a fixed point theorem due to Krasnoselskii and Zabreiko.

Key words: nonlinear three-point boundary value problem; existence; fixed point.
MSC(2000): 34B15
CLC number: O175

1. Introduction

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev[1]. Then Gupta[3] studied three-point boundary value problems for nonlinear ordinary differential equations. Since then, multi-point boundary value problems have received much attention from many authors[4−9]. In particular, Ma and Wang[8] investigated the following more general three-point boundary value problem

$$u''(t)+a(t)u'(t)+b(t)u(t)+h(t)f(u)=0, t\in(0,1), \tag{1}$$

$$u(0)=0, u(1)=\alpha u(\eta) \tag{2}$$

and obtained the existence of a positive solution by using Krasnoselskii's fixed point theorem[2,10] under the conditions that f is either superlinear or sublinear.

Our purpose here is to give a new criteria of the existence to the boundary value problem (1) and (2), and the conditions we need are very easy to verify.

Throughout, we assume that the following conditions hold:

(H1) $a\in C[0,1]$, $b\in C([0,1],(-\infty,0))$;

(H2) $h\in C([0,1],[0,\infty))$ and there exists $x_0\in[0,1]$ such that $h(x_0)>0$;

(H3) $f\in C(R,R)$;

Received date: 2003-06-10
Foundation item: the Natural Science Foundation of Gansu Province of China

(H4) $0<\eta<1$, $0<\alpha\varphi_1(\eta)<1$, here φ_1 is the unique solution of the linear boundary value problem

$$\begin{cases} \varphi_1''(t)+a(t)\varphi_1'(t)+b(t)\varphi_1(t)=0, t\in(0,1), \\ \varphi_1(0)=0, \varphi_1(1)=1. \end{cases} \tag{3}$$

In our arguments, the following well-known fixed point theorem is very crucial.

Theorem 1[11] *Let X be a Banach space, and $F: X\to X$ be completely continuous. Assume that $A: X\to X$ is a bounded linear operator such that 1 is not an eigenvalue of A and*

$$\lim_{\|x\|\to\infty}\frac{\|F(x)-A(x)\|}{\|x\|}=0.$$

Then F has a fixed point in X.

2. Main results

To state the main results of this paper, we need the following two lemmas which was established in [8].

Lemma 1 *Assume that (H1) holds. Let φ_1 be the solution of (3), and φ_2 the solution of*

$$\begin{cases} \varphi_2''(t)+a(t)\varphi_2'(t)+b(t)\varphi_2(t)=0, t\in(0,1), \\ \varphi_2(0)=1, \varphi_2(1)=0, \end{cases} \tag{4}$$

respectively. Then

(i) φ_1 is strictly increasing on $[0,1]$;

(ii) φ_2 is strictly decreasing on $[0,1]$.

Lemma 2 *Assume that (H1) and (H4) hold. Let $y\in C[0,1]$ and u be a solution of the boundary value problem*

$$u''(t)+a(t)u'(t)+b(t)u(t)+y(t)=0, t\in(0,1), \tag{5}$$

$$u(0)=0, u(1)=\alpha u(\eta). \tag{6}$$

Then $u(t)\ge 0$ on $[0,1]$ provided $y\ge 0$.

Corollary 1 *Assume that (H1) and (H4) hold. Let u be a solution of the boundary value problem*

$$u''(t)+a(t)u'(t)+b(t)u(t)=0, t\in(0,1), \tag{7}$$

$$u(0)=0, u(1)=\alpha u(\eta). \tag{8}$$

Then $u(t)\ge 0$ on $[0,1]$.

For convenience, let the Banach space $X=C[0,1]$ be equipped with the norm

$$\|x\|=\max_{t\in[0,1]}|x(t)|$$

and

$$G(t,s)=\frac{1}{\varphi_1'(0)}\begin{cases} \varphi_1(t)\varphi_2(s), & s\ge t, \\ \varphi_1(s)\varphi_2(t), & s\le t. \end{cases}$$

For the function $G(t,s)$, it follows from Lemma 1 that

$$G(t,s) \geq 0, \quad (t,s) \in [0,1] \times [0,1] \tag{9}$$

and

$$G(t,s) > 0, \quad (t,s) \in (0,1) \times (0,1). \tag{10}$$

Our main result is the following theorem.

Theorem 2 *Assume that (H1)–(H4) hold and*

$$\lim_{u \to \infty} \frac{f(u)}{u} = l.$$

If

$$|l| < d = [\max_{t \in [0,1]} \int_0^1 G(t,s)p(s)h(s)\mathrm{d}s + \frac{\alpha}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\mathrm{d}s]^{-1},$$

where $p(t) = \exp(\int_0^t a(s)\mathrm{d}s)$, *then the boundary value problem (1) and (2) has a solution* u^* *and* $u^* \neq 0$ *when* $f(0) \neq 0$.

Proof Suppose that the operator $F: X \to X$ is defined by

$$Fu(t) = \int_0^1 G(t,s)p(s)h(s)f(u(s))\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f(u(s))\mathrm{d}s, \quad t \in [0,1],$$

then it is easy to check that F is completely continuous and that fixed points of F are solutions of the boundary value problem (1) and (2).

We consider the following boundary value problem

$$u''(t) + a(t)u'(t) + b(t)u(t) + lh(t)u(t) = 0, t \in (0,1), \tag{11}$$

$$u(0) = 0, u(1) = \alpha u(\eta). \tag{12}$$

Let the operator $A: X \to X$ be defined by

$$Au(t) = l\left[\int_0^1 G(t,s)p(s)h(s)u(s)\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)u(s)\mathrm{d}s\right], \quad t \in [0,1],$$

then it is easy to know that A is completely continuous (so bounded) and linear, and that fixed points of A are solutions of the boundary value problem (11) and (12).

First, we claim that 1 is not an eigenvalue of A.

We consider two cases:

Case (i) Suppose $l = 0$ and u is a solution of the boundary value problem (11) and (12), then it is easy to verify that $-u$ is also a solution of the boundary value problem (11) and (12). So it follows from Corollary 1 that $u(t) \geq 0$ and $-u(t) \geq 0$ on $[0,1]$, i.e., $u(t) \equiv 0$ on $[0,1]$. This shows that the boundary value problem (11) and (12) does not have any nontrivial solution.

Case (ii) Suppose $l \neq 0$. If the boundary value problem (11) and (12) has a nontrival solution u, then $\|u\| > 0$. Noting (9), (10) and Lemma 1, we have

$$\begin{aligned}
\|u\| &= \|Au\| \\
&= \max_{t\in[0,1]} \left| l \left[\int_0^1 G(t,s)p(s)h(s)u(s)\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)u(s)\mathrm{d}s \right] \right| \\
&= |l| \max_{t\in[0,1]} \left| \int_0^1 G(t,s)p(s)h(s)u(s)\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)u(s)\mathrm{d}s \right| \\
&\le |l| \max_{t\in[0,1]} \left[\left| \int_0^1 G(t,s)p(s)h(s)u(s)\mathrm{d}s \right| + \left| \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)u(s)\mathrm{d}s \right| \right] \\
&\le |l| \max_{t\in[0,1]} \left[\int_0^1 G(t,s)p(s)h(s)\left|u(s)\right|\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\left|u(s)\right|\mathrm{d}s \right] \\
&\le |l|\,\|u\| \max_{t\in[0,1]} \left[\int_0^1 G(t,s)p(s)h(s)\mathrm{d}s + \frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\mathrm{d}s \right] \\
&\le |l|\,\|u\| \left[\max_{t\in[0,1]} \int_0^1 G(t,s)p(s)h(s)\mathrm{d}s + \frac{\alpha}{1-\alpha\varphi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)\mathrm{d}s \right] \\
&< d\,\|u\|\,\frac{1}{d} = \|u\|,
\end{aligned}$$

which is impossible. Hence, the boundary value problem (11) and (12) does not have any nontrivial solution, i.e., 1 is not an eigenvalue of A.

Next, we will prove that

$$\lim_{\|u\|\to\infty} \frac{\|F(u)-A(u)\|}{\|u\|} = 0.$$

Since $\lim_{u\to\infty} \frac{f(u)}{u} = l$, then for $\forall \varepsilon > 0$, there must exist $M_1 > 0$ such that

$$|f(u) - lu| < \varepsilon\,|u|, \quad |u| > M_1. \tag{13}$$

Let $M_2 = \max_{|u|\le M_1} |f(u)|$. We can choose $L > M_1$ such that

$$\frac{M_2 + |l|\,M_1}{L} < \varepsilon,$$

then for $\forall u \in X$ and $\|u\| > L$,

(i) If $s \in [0,1]$ and $|u(s)| \le M_1$, then

$$|f(u(s)) - lu(s)| \le |f(u(s))| + |l|\,|u(s)| \le M_2 + |l|\,M_1 < \varepsilon L < \varepsilon\,\|u\|.$$

(ii) If $s \in [0,1]$ and $|u(s)| > M_1$, then from (13), we have

$$|f(u(s)) - lu(s)| \le \varepsilon\,|u(s)| \le \varepsilon\,\|u\|.$$

Hence,

$$|f(u(s)) - lu(s)| \le \varepsilon\,\|u\|, \forall s \in [0,1]. \tag{14}$$

So,

$$
\begin{aligned}
&\|F(u)-A(u)\| \\
&= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)p(s)h(s)\left[f(u(s))-lu(s)\right]\mathrm{d}s+\right. \\
&\qquad\left.\frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)\left[f(u(s))-lu(s)\right]\mathrm{d}s\right| \\
&\leq \max_{t\in[0,1]}\left[\int_0^1 G(t,s)p(s)h(s)\left|f(u(s))-lu(s)\right|\mathrm{d}s+\right. \\
&\qquad\left.\frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)\left|f(u(s))-lu(s)\right|\mathrm{d}s\right] \\
&\leq \varepsilon\|u\|\max_{t\in[0,1]}\left[\int_0^1 G(t,s)p(s)h(s)\mathrm{d}s+\frac{\alpha\varphi_1(t)}{1-\alpha\varphi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)\mathrm{d}s\right] \\
&\leq \varepsilon\|u\|\left[\max_{t\in[0,1]}\int_0^1 G(t,s)p(s)h(s)\mathrm{d}s+\frac{\alpha}{1-\alpha\varphi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)\mathrm{d}s\right] \\
&= \frac{\varepsilon}{d}\|u\|
\end{aligned}
$$

implies that

$$\lim_{\|u\|\to\infty}\frac{\|F(u)-A(u)\|}{\|u\|}=0.$$

It follows from Theorem 1 that F has a fixed point $u^*\in X$, i.e., u^* is a solution of the boundary value problem (1) and (2), and it is obvious that $u^*\neq 0$ when $f(0)\neq 0$.

References:

[1] Il'in V A, MOISEEV E I. *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator* [J]. Differential Equations, 1987, **23**: 979–987.

[2] GUO Da-jun, LAKSHMIKANTHAM V. *Nonlinear problems in Abstract Cones* [M]. Notes and Reports in Mathematics in Science and Engineering, 5. Academic Press, Inc., Boston, MA, 1988.

[3] GUPTA C P. *Solvability of a three-point nonlinear boundary value problems for a second order ordinary differential equation* [J]. J. Math. Anal. Appl., 1992, **168**: 540–551.

[4] GUPTA C P. *A sharp condition for the solvability of a three-point second order boundary value problem* [J]. J. Math. Anal. Appl., 1997, **205**: 579–586.

[5] C. P. Gupta and S.Trofimchuk. Existence of a solution to a three-point boundary value problem and the spectral radius of a related linear operator [J]. Nonlinear Anal., 1998, **34**: 498–507.

[6] MA Ru-yun. *Existence theorems for a second order three-point boundary value problem* [J]. J. Math. Anal. Appl., 1997, **212**: 430–442.

[7] MA Ru-yun. *Positive solutions of a nonlinear three-point boundary value problem* [J]. E. J. Differential Equations, 1999, **34**: 1–8.

[8] MA Ru-yun, WANG Hai-yan. *Positive solutions of nonlinear three-point boundary-value problems* [J]. J. Math. Anal. Appl., 2003, **279**(1): 216–227.

[9] SUN Jian-ping, HUO Hai-feng, LI Wan-tong. *Positive solutions to singular nonlinear three point boundary value problems* [J]. Dynam. Systems Appl., 2003, **12**: 275–283.

[10] KRASNOSELSKII M A. *Positive Solution of Operator Equations* [M]. Noordhoff, Groningen, 1964.

[11] KRASNOSELSKII M A, ZABREIKO P P. *Geometrical Methods of Nonlinear Analysis* [M]. Spriger-Verlag, New York, 1984.

非线性三点边值问题解的一个新的存在定理

孙建平， 赵亚红
(兰州理工大学应用数学系，甘肃 兰州 730050)

摘要：利用 Krasnoselskii-Zabreiko 不动点定理获得了非线性三点边值问题

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+h(t)f(u)=0, t\in(0,1) \\ u(0)=0, u(1)=\alpha u(\eta) \end{cases}$$

解的一个新的存在定理.

关键词：非线性三点边值问题；存在性；不动点.